1. Practice with Set Operations

In these questions, let S be the set $\{3, 2, 1, 0\}$ and let T be the set $\{1, \{1, 2\}, 4, 6\}$.

a. $\in$ vs. $\subseteq$

   (1) Is $1 \in S$? Is $1 \subseteq S$?

   (2) Is $\{1, 2\} \in S$? Is $\{1, 2\} \subseteq S$? Is $\{1, 2\} \in T$? Is $\{1, 2\} \subseteq T$?

   (3) Is $2 \in T$? Is $\{1\} \in T$? Is $\{1\} \subseteq T$?

   (4) Is $S \subseteq \mathbb{N}$? Is $T \subseteq \mathbb{N}$? (In this class, we treat 0 as a natural number.)

b. Cardinality

   (1) Is $|S|$ equal to $|T|$?

   (2) $\aleph_0$ is the cardinality of $\mathbb{N}$, the set of natural numbers. What is $|\{\mathbb{N}\}|$?

   (3) What is $|\{n \in \mathbb{N} \mid n < 103\}|$? Describe this set in words.

   (4) What is $|\{n \in \mathbb{N} \mid n \geq 103\}|$? (Try writing out the set. If you can find a way to pair each element of this set with each element of another set, the sets have the same cardinality.)

c. Sets containing the empty set

   (1) Is $\varnothing \in \varnothing$? Is $\varnothing \subseteq \varnothing$?

   (2) Is $\varnothing \in \{\varnothing\}$? Is $\varnothing \in \big\{\{\varnothing\}\big\}$?

   (3) Describe the set $\{\varnothing, \{\varnothing\}\}$ in words. What set is it the power set of?

2. Power Sets

a. What is the power set of the following sets?

   (1) $\{103\}$

   (2) $\{103, 106\}$

   (3) $\{\{103, 106\}\}$ (Hint: What is the cardinality of this set?)

   (4) $\wp(\{103\})$

b. Based on your answers above, for a finite set S, what is the relationship between $|S|$ and $|\wp(S)|$? If you don't see a pattern yet, try taking the power set of a set with 3 or 4 elements.

c. Give three examples of **elements** of the set $\wp(\mathbb{N})$. These are **subsets** of the set __________.

d. Give three examples of **elements** of the set $\wp(\wp(\mathbb{N}))$. These are **subsets** of the set __________.

3. Direct Proofs of Equations

A **Pythagorean triple** is an ordered trio of positive natural numbers (a, b, c) that have the property that $a^2 + b^2 = c^2$. For example, $(3, 4, 5)$ is a Pythagorean triple because $3^2 + 4^2 = 5^2$. Same with $(8, 15, 17)$ and $(5, 12, 13)$. Meanwhile, $(1, 2, 3)$ and $(5, 4, 3)$ aren't Pythagorean triples.

a. We'll walk through the process of writing a direct proof of this statement: For any Pythagorean triple (x, y, z), if x is odd and y is even, then z^2 is odd.

- Is this whole statement universally quantified (a claim about everything from a certain category) or existentially quantified (a claim about a specific example)?
- Based on the answer to the previous question, should you (the proof writer) give specific values for the variables x, y, and z, or should the proof reader be able to pick whatever values they want?
- What is the implication in this statement? What are the antecedent and consequent?
- Based on the answer to the previous question, what will we assume and what will we want-to-show as part of this proof? Expand these using the formal definition of even and odd numbers. (This statement can be proven without using the formal definitions, using theorems mentioned in class — but let's practice applying the definitions.)
- Since the consequent is an existential statement, you, the proof writer, must supply the value for the variable. Combine the equations you have to find what value will work.
- Now, write the formal proof.

b. We'll walk through the process of writing a direct proof of this statement: For any positive natural numbers $n, x, y,$ and z, if (x, y, z) is a Pythagorean triple, then (nx, ny, nz) is a Pythagorean triple. Start by answering these questions:

- Is this whole statement universally or existentially quantified? How should the values of the variables n, x, y, and z be determined?
- What is the implication in this statement? What are the antecedent and consequent?
- What will we assume and what will we want-to-show as part of this proof?

Assume:

$$\underline{\qquad\qquad} + \underline{\qquad\qquad} = \underline{\qquad\qquad}$$

Want to show:

$$\underline{\qquad\qquad} + \underline{\qquad\qquad} = \underline{\qquad\qquad}$$

- Now, prove the statement. (**Key tip:** when your want-to-show is an equation, always start with one side of the equation and manipulate it to find the other.)

4. Negation and Proof by Contradiction

Prove this theorem by **contradiction**: For any Pythagorean triple (a, b, c), $(a+1, b+1, c+1)$ is *not* a Pythagorean triple. Answer these questions first:

- What is the negation of this theorem? (Hint: how does a universal statement change when you take the negation?)
- To write a proof by contradiction, what will we assume? You should have two equations; expand them out.
- How can we combine these equations to lead to something that cannot be true? Only begin to write the proof once you're solid on what the contradiction actually is.

Key tip: The values of the variables here are "existentially picked", which is a third way of introducing variables in proofs, used when we know something exists but don't know what it is. As a result, we can't give specific numbers as the proof writer (since we don't know what it is), and the reader can't pick whatever values they want (since the values have to have certain properties).

5. Proof by Contrapositive

Prove this theorem by **contrapositive**: For any Pythagorean triple (a, b, c), if a^2 is even, then $c \neq b + 1$.

a. First, let's figure out what the contrapositive is.
    - What are the antecedent and consequent of the implication in the theorem?
    - What is the contrapositive of the implication?

b. Then, let's figure out the proof structure that the contrapositive suggests.
    - What are the antecedent and consequent of the contrapositive? What do we assume and what do we want to show?
    - The entire theorem is universally quantified. Should the reader pick whatever values they want, or should you, the proof writer, give specific values?

c. You should have two equations in the assumptions. How can we combine those equations to lead toward our want-to-show?

    Key tip: Once you've figured out the math and are starting to write the proof, notice that we are trying to prove something about a^2. You'll want to start with the quantity a^2 and then manipulate it to get the desired result. When presenting a math proof, you should always present true statements, so you should not start with the equation you want to show.