1. Practice with Set Operations

In these questions, let S be the set $\{3, 2, 1, 0\}$ and let T be the set $\{1, \{1, 2\}, 4, 6\}$.

a. $\in$ vs. $\subseteq$

(1) Is $1 \in S$? Is $1 \subseteq S$?

> Yes, 1 is one of the elements in the set $\{0, 1, 2, 3\}$.
>
> No, 1 is not a subset of S: 1 is a number, not a set, so it can't be a subset of anything.

(2) Is $\{1, 2\} \in S$? Is $\{1, 2\} \subseteq S$? Is $\{1, 2\} \in T$? Is $\{1, 2\} \subseteq T$?

> No, $\{1, 2\}$ is not an element in S: S only contains the four elements 0, 1, 2, 3, which are numbers, not sets.
>
> Yes, $\{1, 2\}$ is a subset of S: both of its elements are also an element in S. Sets are unordered, so it doesn't matter that these numbers show up in a different order here than in how I wrote S.
>
> Yes, $\{1, 2\}$ is an element of T, since it is listed as one of the elements.
>
> No, $\{1, 2\}$ is not a subset of T. The elements of $\{1, 2\}$ are 1 and 2. Even though 1 is an element in T, 2 is not an element in T.

(3) Is $2 \in T$? Is $\{1\} \in T$? Is $\{1\} \subseteq T$?

> No, 2 is not one of the elements in T. Even though T includes the set $\{1, 2\}$ as an element, the number 2 is not an element of T. Only the numbers 1, 4, and 6 are.
>
> No, $\{1\}$ is also not one of the elements in T. The set $\{1, 2\}$ is not the same as the set $\{1\}$.
>
> Yes, $\{1\}$ is a subset of T. The set $\{1\}$ contains one element, 1. 1 is also an element in T.

(4) Is $S \subseteq \mathbb{N}$? Is $T \subseteq \mathbb{N}$? (In this class, we treat 0 as a natural number.)

> Yes, S is a subset of $\mathbb{N}$. To check this, we can see that every element in S is a natural number, since the natural numbers start at 0.
>
> No, T is not a subset of $\mathbb{N}$. There is an element in T that is not also in $\mathbb{N}$, namely the set $\{1, 2\}$. All of the elements in $\mathbb{N}$ are numbers, so the set $\{1, 2\}$ is not an element in $\mathbb{N}$.

b. Cardinality

(1) Is $|S|$ equal to $|T|$?

> Yes, $|S| = |T|$. S has four elements, which are all numbers, and T has four elements: three numbers (1, 4, and 6) and one set $\{1, 2\}$.

(2) $\aleph_0$ is the cardinality of $\mathbb{N}$, the set of natural numbers. What is $|\{\mathbb{N}\}|$?

> The cardinality of this set is 1. There is only one element in the set, namely the set $\mathbb{N}$. It doesn't matter how many elements are in the set $\mathbb{N}$, because it just counts as one element.

(3) What is $|\{n \in \mathbb{N} \mid n < 103\}|$? Describe this set in words.

> The cardinality of this set is 103. In words, we could describe $|\{n \in \mathbb{N} \mid n < 103\}|$ as "the set of all natural numbers less than 103." Since 0 is a natural number, this set could also be written as $\{0, 1, 2, ..., 102\}$. There are 103 elements in this set.

(4) What is $|\{n \in \mathbb{N} \mid n \geq 103\}|$? (Try writing out the set. If you can find a way to pair each element of this set with each element of another set, the sets have the same cardinality.)

> The cardinality of this set is actually also $\aleph_0$. We can pair off any element n from this set with the element $n - 103$ from $\mathbb{N}$.

c. Sets containing the empty set

(1) Is $\varnothing \in \varnothing$? Is $\varnothing \subseteq \varnothing$?

> No, $\varnothing$ is not an element in $\varnothing$: $\varnothing$ contains no elements, so nothing can be an element of it.
>
> Yes, $\varnothing$ is a subset of $\varnothing$. There are two ways we can see this. First, the empty set is a subset of every set. Also, every set is a subset of itself.

(2) Is $\varnothing \in \{\varnothing\}$? Is $\varnothing \in \big\{\{\varnothing\}\big\}$?

> Yes, it's the case that $\varnothing \in \{\varnothing\}$! The set $\{\varnothing\}$ has one element, which is the empty set.
>
> No, it's the case that $\varnothing \notin \big\{\{\varnothing\}\big\}$. The set $\big\{\{\varnothing\}\big\}$ has one element, which is the set containing the empty set. As we found in lecture, the set containing the empty set is not the same as the empty set itself!

(3) Describe the set $\{\varnothing, \{\varnothing\}\}$ in words. What set is it the power set of?

This set could be described as "the set containing two elements: the empty set and the set containing the empty set".

This set is the power set of $\{\varnothing\}$.

2. Power Sets

a. What is the power set of the following sets?

(1) $\{103\}$

> $\wp(\{103\})$ is the power set of $\{103\}$, or the set of all of the subsets of $\{103\}$. The set $\{103\}$ has two subsets: the empty set $\varnothing$, which is a subset of every set, and the set $\{103\}$, since every set is a subset of itself. There are no other subsets of this set. Therefore, the power set of $\{103\}$ is $\big\{\varnothing, \{103\}\big\}$.

(2) $\{103, 106\}$

> There are four subsets of this set: we can include either element, or both elements, or neither. So $\wp(\{103, 106\})$ is $\big\{\varnothing, \{103\}, \{106\}, \{103, 106\}\big\}$.

(3) $\{\{103, 106\}\}$ (Hint: What is the cardinality of this set?)

> First, notice that the cardinality of this set is 1. There is only one element in the set: that element itself is the set $\{103, 106\}$. As a result, the solution is similar to the previous part of this problem: the set has two subsets, the empty set $\varnothing$, which is a subset of every set, and the set $\{\{103, 106\}\}$, since every set is a subset of itself. There are no other subsets of this set. Therefore, the power set of $\{\{103, 106\}\}$ is $\big\{\varnothing, \{\{103, 106\}\}\big\}$.

(4) $\wp(\{103\})$

> $\wp(\wp(\{103\}))$ is the power set of the set that we found: the set of all the subsets of $\{\varnothing, \{103\}\}$. There are two elements in this set, and so there are four subsets of the set: the set that doesn't contain either element (so the empty set), the set that contains $\varnothing$ but doesn't contain $\{103\}$, the set that contains $\{103\}$ but doesn't contain $\varnothing$, and the set that contains both. Therefore, the power set of the power set of $\{103\}$ is: $\Big\{\varnothing, \{\varnothing\}, \big\{\{103\}\big\}, \big\{\varnothing, \{103\}\big\}\Big\}$. Whew!

b. Based on your answers above, for a finite set S, what is the relationship between $|S|$ and $|\wp(S)|$? If you don't see a pattern yet, try taking the power set of a set with 3 or 4 elements.

> From the previous part, we see that when $|S| = 0$, $|\wp(S)| = 1$; when $|S| = 1$, $|\wp(S)| = 2$; and when $|S| = 2$, $|\wp(S)| = 4$. For larger sets, you can see that when $|S| = 3$, $|\wp(S)| = 8$, and when $|S| = 4$, $|\wp(S)| = 16$. The pattern here is that the number of power sets $|\wp(S)|$

is always equal to $2^{|S|}$.

(Beyond the scope of this class, but discussed more in CS109: The intuition behind this is that every subset is a way of picking elements from the set. There are 2 options for every element from the set: either it's included in the subset or it's not. With 2 choices per element, the total number of options is 2 raised to the power of the number of elements in the set.)

c. Give three examples of **elements** of the set $\wp(\mathbb{N})$. These are **subsets** of the set __________.

Elements of $\wp(\mathbb{N})$ are subsets of $\mathbb{N}$, aka sets with elements that are natural numbers. Some examples are $\varnothing$, $\{1\}$, $\{103, 106\}$, the set of even natural numbers, and $\mathbb{N}$.

d. Give three examples of **elements** of the set $\wp(\wp(\mathbb{N}))$. These are **subsets** of the set __________.

Elements of $\wp(\wp(\mathbb{N}))$ are subsets of $\wp(\mathbb{N})$. Based on the definition of a subset, any subset of $\wp(\mathbb{N})$ must have all of its elements also be elements of $\wp(\mathbb{N})$. The elements of $\wp(\mathbb{N})$ are sets containing natural numbers. This means any elements of $\wp(\wp(\mathbb{N}))$ must be sets containing sets of natural numbers.

Some examples are:

- $\{\varnothing\}$
- $\{\{1\}, \{103, 104, 105\}\}$
- $\{\mathbb{N}\}$ (We saw earlier that this is different from $\mathbb{N}$.)
- $\wp(\mathbb{N})$

3. Direct Proofs of Equations

A **Pythagorean triple** is an ordered trio of positive natural numbers (a, b, c) that have the property that $a^2 + b^2 = c^2$. For example, $(3, 4, 5)$ is a Pythagorean triple because $3^2 + 4^2 = 5^2$. Same with $(8, 15, 17)$ and $(5, 12, 13)$. Meanwhile, $(1, 2, 3)$ and $(5, 4, 3)$ aren't Pythagorean triples.

a. We'll walk through the process of writing a direct proof of this statement: For any Pythagorean triple (x, y, z), if x is odd and y is even, then z^2 is odd.

- Is this whole statement universally quantified (a claim about everything from a certain category) or existentially quantified (a claim about a specific example)?

  This is a **universally quantified** statement, describing something about all Pythagorean triples, rather than a fact about one particular Pythagorean triple. One way we can tell is because of the "for any".

- Based on the answer to the previous question, should you (the proof writer) give specific values for the variables x, y, and z, or should the proof reader be able to pick whatever values they want?

  The reader must be in control. The reader can pick any values they want to fill in for the x, y, and z. This is because a proof of a universal statement needs to work for whatever values are used.

- What is the implication in this statement? What are the antecedent and consequent?

  The implication is "if x is odd and y is even, then z^2 is odd."

  The antecedent is "x is odd and y is even". The consequent is "z^2 is odd".

- Based on the answer to the previous question, what will we assume and what will we want-to-show as part of this proof? Expand these using the formal definition of even and odd numbers. (This statement can be proven without using the formal definitions, using theorems mentioned in class — but let's practice applying the definitions.)

  We assume the antecedent: x is odd and y is even. We want to show the consequent: we want to show that z^2 is odd.

  To expand these, we can assume that there is an integer a where $x = 2a + 1$ and there is an integer b where $y = 2b$. We would like to show that there is an integer c where $z^2 = 2c + 1$. Notice that we are using three different letters from the letters used in the definition of even and odd, because these numbers, a, b, and c, may have different values from each other.

- Since the consequent is an existential statement, you, the proof writer, must supply the value for the variable. Combine the equations you have to find what value will work.

  We know from the definition of a Pythagorean triple that $z^2 = x^2 + y^2$. We also know that $x = 2a + 1$ and $y = 2b$. We can combine these two facts to see that

  $$\begin{aligned} z^2 &= (2a+1)^2 + (2b)^2 \\ &= 4a^2 + 4a + 1 + 4b^2 \\ &= 2(2a^2 + 2b^2 + 2a) + 1. \end{aligned}$$

  This means that the value of c is $2a^2 + 2b^2 + 2a$.

- Now, write the formal proof.

Theorem: For any Pythagorean triple (x, y, z), if x is odd and y is even, then z^2 is odd.

Proof: Pick any Pythagorean triple (x, y, z) where x is odd and y is even. We will show that z^2 is odd.

Since x is odd, there is an integer a where $x = 2a + 1$. Since y is even, there is an integer b where $y = 2b$. Since (x, y, z) is a Pythagorean triple, we know that

$$\begin{aligned} z^2 &= x^2 + y^2 \\ &= (2a+1)^2 + (2b)^2 \\ &= 4a^2 + 4a + 1 + 4b^2 \\ &= 2(2a^2 + 2b^2 + 2a) + 1. \end{aligned}$$

We can see that there is an integer c (namely, $2a^2 + 2b^2 + 2a$) such that $z^2 = 2c + 1$. Therefore, we see that z^2 is odd, as required. ■

Here's an alternative proof using facts mentioned in class:

Proof: Pick any Pythagorean triple (x, y, z) where x is odd and y is even. We will show that z^2 is odd.

Since x is odd and the product of two odd numbers is odd, x^2 is odd. Since y is even and the product of two even numbers is even, y^2 is even. Since (x, y, z) is a Pythagorean triple, we know that z^2 is the sum of an odd number, x^2, and an even number, y^2. Because the sum of an odd number and an even number is odd, we see that z^2 is odd, as required. ■

b. We'll walk through the process of writing a direct proof of this statement: For any positive natural numbers n, x, y, and z, if (x, y, z) is a Pythagorean triple, then (nx, ny, nz) is a Pythagorean triple. Start by answering these questions:

- Is this whole statement universally or existentially quantified? How should the values of the variables n, x, y, and z be determined?

  This is a **universally quantified** statement, describing something about all natural numbers, rather than a fact about one particular natural number. One way we can tell is because of the "for any".

  The reader must be in control. The reader can pick any values they want to fill in for the n, x, y, and z. This is because a proof of a universal statement needs to work for whatever values are used.

- What is the implication in this statement? What are the antecedent and consequent?

  The implication is "if (x, y, z) is a Pythagorean triple, then (nx, ny, nz) is a Pythagorean triple".

  The antecedent is "(x, y, z) is a Pythagorean triple" and the consequent is "(nx, ny, nz) is a Pythagorean triple".

- What will we assume and what will we want-to-show as part of this proof?

  When proving an implication, we want to assume the antecedent and we want to show that the consequent is true. Here they are as equations:

  Assume:
  $$x^2 + y^2 = z^2$$

  Want to show:
  $$(nx)^2 + (ny)^2 = (nz)^2$$

- Now, prove the statement. (**Key tip:** when your want-to-show is an equation, always start with one side of the equation and manipulate it to find the other.)

Theorem: For any natural number n, if (x, y, z) is a Pythagorean triple, then (nx, ny, nz) is a Pythagorean triple.

Proof: Pick any natural numbers n, x, y, and z where (x, y, z) is a Pythagorean triple. We will show that (nx, ny, nz) is also a Pythagorean triple.

We can see that

$$\begin{aligned}(nx)^2 + (ny)^2 &= n^2x^2 + n^2y^2 \\ &= n^2(x^2 + y^2).\end{aligned}$$

Since we know (x, y, z) is a Pythagorean triple, we can see that

$$\begin{aligned}n^2(x^2 + y^2) &= n^2z^2 \\ &= (nz)^2.\end{aligned}$$

Therefore, (nx, ny, nz) is a Pythagorean triple, as required. ■

4. Negation and Proof by Contradiction

Prove this theorem by **contradiction**: For any Pythagorean triple (a, b, c), $(a + 1, b + 1, c + 1)$ is *not* a Pythagorean triple. Answer these questions first:

- What is the negation of this theorem? (Hint: how does a universal statement change when you take the negation?)

  The negation of the universal statement "for all P, Q is true" is "there exists a P where Q is not true."

  In this case, P is "Pythagorean triples (a, b, c)" and Q is "$(a + 1, b + 1, c + 1)$ is not a Pythagorean triple".

  So the negation of the theorem is "There is a Pythagorean triple (a, b, c) where $(a + 1, b + 1, c + 1)$ *is* a Pythagorean triple."

- To write a proof by contradiction, what will we assume? You should have two equations; expand them out.

  We want to assume the negation of the theorem: "there is a Pythagorean triple (a, b, c) where $(a + 1, b + 1, c + 1)$ *is* a Pythagorean triple."

  To write this as equations, we have that $a^2 + b^2 = c^2$ and $(a + 1)^2 + (b + 1)^2 = (c + 1)^2$.

- How can we combine these equations to lead to something that cannot be true? Only begin to write the proof once you're solid on what the contradiction actually is.

  How to approach this proof: Again, let's write down the facts that we know. In contrast to when we're doing a direct proof, we want to work toward something that isn't true.

  - (a, b, c) is a Pythagorean triple, meaning that $a^2 + b^2 = c^2$
  - $(a+1, b+1, c+1)$ is a Pythagorean triple, meaning that $(a+1)^2+(b+1)^2 = (c+1)^2$

  Since these equations involve the same variables, it's a good idea to simplify them and see if they could be combined. The second equation works out to:

  $$a^2 + 2a + 1 + b^2 + 2b + 1 = c^2 + 2c + 1$$

  Now we can combine these two equations! Since we know that $a^2 + b^2 = c^2$, we arrive at this equation:

  $$2a + 2b + 2 = 2c + 1$$

It might be easier to see what the contradiction is if we write it like this:

$$2(a + b + 1) = 2c + 1$$

The left hand side of this equation is even, but the right hand side is odd. No number can be both even and odd, so we're done!

Theorem: For any Pythagorean triple (a, b, c), $(a+1, b+1, c+1)$ is not a Pythagorean triple.

Proof: Assume for the sake of contradiction that there are positive integers a, b, and c such that (a, b, c) is a Pythagorean triple and $(a + 1, b + 1, c + 1)$ is also a Pythagorean triple. Expanding out the definition of a Pythagorean triple, we see that

$$(a + 1)^2 + (b + 1)^2 = (c + 1)^2.$$

Expanding and simplifying this equation, we see that

$$a^2 + 2a + 1 + b^2 + 2b + 1 = c^2 + 2c + 1$$

and cancelling out $a^2 + b^2$ with c^2, because (a, b, c) is a Pythagorean triple, we see that

$$2a + 2b + 2 = 2c + 1.$$

This equation simplifies to

$$2(a + b + 1) = 2c + 1.$$

However, this is impossible: the left-hand side is an even number, and the right-hand side is an odd number. Since no number can be both even and odd, we have reached a contradiction, so our assumption must have been wrong. Therefore, we have shown that for any Pythagorean triple (a, b, c), $(a + 1, b + 1, c + 1)$ must not be a Pythagorean triple, as required. ■

Key tip: The values of the variables here are "existentially picked", which is a third way of introducing variables in proofs, used when we know something exists but don't know what it is. As a result, we can't give specific numbers as the proof writer (since we don't know what it is), and the reader can't pick whatever values they want (since the values have to have certain properties).

5. Proof by Contrapositive

Prove this theorem by **contrapositive**: For any Pythagorean triple (a, b, c), if a^2 is even, then $c \neq b + 1$.

a. First, let's figure out what the contrapositive is.

- What are the antecedent and consequent of the implication in the theorem?

  The antecedent is "a^2 is even" and the consequent is "$c \neq b + 1$".

- What is the contrapositive of the implication?

  First, let's take the negation of the antecedent and consequent: respectively, "a^2 is odd" and "$c = b + 1$".

  Then, we can see the contrapositive is "if $c = b + 1$, a^2 is odd".

b. Then, let's figure out the proof structure that the contrapositive suggests.

- What are the antecedent and consequent of the contrapositive? What do we assume and what do we want to show?
- The entire theorem is universally quantified. Should the reader pick whatever values they want, or should you, the proof writer, give specific values?

  The antecedent is "$c = b + 1$", and the consequent is "a^2 is odd".

  We would have to assume that $c = b + 1$, and show that a^2 is odd. Since the entire statement is universally quantified, we should have the reader pick any arbitrary Pythagorean triple (a, b, c), where $c = b + 1$.

c. You should have two equations in the assumptions. How can we combine those equations to lead toward our want-to-show?

Key tip: Once you've figured out the math and are starting to write the proof, notice that we are trying to prove something about a^2. You'll want to start with the quantity a^2 and then manipulate it to get the desired result. When presenting a math proof, you should always present true statements, so you should not start with the equation you want to show.

How to approach this proof: One useful strategy is to write down everything we know or are assuming and see if we can combine any of the facts:

- a, b, c is a Pythagorean triple, so $a^2 + b^2 = c^2$

- $c = b + 1$

Since we know what c is, let's try substituting it into the first equation, which results in $a^2 + b^2 = (b+1)^2 = b^2 + 2b + 1$.

We can then cancel out the b^2 terms on each side, and realize that this simplifies to $a^2 = 2b + 1$! This means that a^2 is odd. We will use a modified version of this approach in the final proof, keeping in mind that when you want to prove something about a specific quantity, starting with that quantity can make the proof clearer.

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Here's an outline of another strategy. I think this strategy uses more polynomial manipulation than you will be using in the rest of the course, so I've written up the previous strategy.

Again, since we want to prove something about a, it would be useful to solve for a^2 in the above equation, yielding that $a^2 = c^2 - b^2$. The right-hand side of this equation can be factored into $(c-b)(c+b)$. We can substitute in 1 for $c - b$ to find that $a^2 = c + b$. Then, we have two paths forward. One approach would be to substitute $c = b + 1$ in directly to find that $a^2 = 2b + 1$ as before. Alternatively, knowing that $c = b + 1$, we can show that if b is even then c is odd, and if b is odd then c is even. Because exactly one of the two is odd, their sum $c + b$ is odd, so a^2 must be odd as well.

Theorem: For all Pythagorean triples (a, b, c), if a^2 is even, then $c \neq b + 1$.

Proof: We will prove the contrapositive of this statement, namely: for all Pythagorean triples (a, b, c), if $c = b + 1$, then a^2 is odd. To do so, pick an arbitrary Pythagorean triple (a, b, c) where $c = b + 1$. We will show that a^2 is odd.

Because (a, b, c) is a Pythagorean triple and $c = b + 1$, we can see that

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= (b+1)^2 - b^2 \\ &= b^2 + 2b + 1 - b^2 \\ &= 2b + 1 \end{aligned}$$

This means that there is an integer, namely b, where $a^2 = 2b + 1$, so a^2 is odd, which is what we wanted to show. ■