1. Set Theory

1.1 Set Theory Symbols

In our first lecture, we introduced a bunch of symbols in the context of set theory. As a reference, here's a list of all of the symbols we encountered:

$$\begin{array}{cccc} \in & \notin & \subseteq & \not\subseteq \\ \varnothing & \mathbb{N} & \mathbb{Z} & \mathbb{R} \\ \cup & \cap & \setminus & \Delta \\ \wp(\cdot) & |\cdot| & \aleph_0 & \end{array}$$

(Assume $\cdot$ is where the name of a set goes.)

What do each of these symbols mean? Give an example of how each might be used.

1.2 Set Theory in Real Life: Pets and Sets

This problem practices using set operations and set-builder notation to express certain ideas. Say A is the set of all animals, B is the set of all brown things, C is the set of all cats, and D is the set of all dogs.

a. Write mathematical statements equivalent to these statements by using some of the sets A, B, C, D and some of the set theory symbols we've learned about:

   (1) "All cats are animals."

   (2) "The number of brown cats is the same as the number of brown dogs."

   (3) "There isn't anything that is both a cat and a dog."

b. Use set-builder notation and some of the sets A, B, C, D to express these sets:

   (1) The set of cute dogs

   (2) The set containing everything that isn't a cat or a dog

1.3 Set-Builder Notation

Consider the following set:

$$S = \{n \in \mathbb{N} \mid n \text{ is odd}\}$$

Here's a mathematical argument. Is it correct? Why or why not?

> Let's pick the number $n = 137$. We know that n is odd, because 137 doesn't cleanly divide by two. We also know that n is a natural number, because it's a whole number and isn't negative. Since n is odd and $n \in \mathbb{N}$, $S = \{137\}$.

1.4 Exploring Sets

You'll probably need to play around with these properties a bit before you find something that works, so try things out and see what you come up with!

a. Find sets A and B where...

   (1) $A \notin B$, but $A \subseteq B$.

   (2) $A \in B$, but $A \not\subseteq B$.

   (3) $A \in B$ and $A \subseteq B$.

b. Find a set A where...

   (1) $A \in \wp(A)$.

   (2) $A \subseteq \wp(A)$.

2. Mathematical Proofs

2.1 Properties of Odd and Even Numbers

a. Prove this statement: for any integer n, if n is even, then $n - 1$ is odd.

b. What is the smallest even natural number?

2.2 Proof by Contradiction: Pythagorean Triples

As a refresher, a Pythagorean triple is an ordered trio of positive natural numbers (a, b, c) where $a^2 + b^2 = c^2$.

Prove this theorem by **contradiction**: For any Pythagorean triple (a, b, c), if a^2 is even, then $c \neq b + 1$. (Hint: How does an implication change when you take the negation?)

2.3 Proof by Contrapositive: Multiples of Four

In lecture, we proved that if n is an integer, then n is even if and only if n^2 is even. This question explores some other properties about the relationship between n and n^2, giving you a chance to practice with proofs and proof techniques.

An integer n is called a **multiple of four** if n is equal to $4k$ for some integer k. Consider this statement:

$$\text{For any integer } n \text{, if } n^2 \text{ is not a multiple of four, then } n \text{ is odd.} \qquad (\bigstar)$$

a. Proof by contrapositive

   (1) What is the contrapositive of the implication in statement ($\bigstar$)?

(2) Prove statement (★) using a proof by contrapositive.

b. Prove that if n is odd, there exists an integer k where $n^2 = 4k + 1$.

In combination with the contrapositive of statement (★), we can see that every perfect square is either a multiple of four or 1 greater than a multiple of four. Pretty cool!

2.4 Proving Mixed Statements

I recommend trying these problems after the problem set 1 question on modular congruence!

Here's a refresher on modular congruence: For an integer k, we say that $a \equiv_k b$ when there exists an integer q such that $a = b + kq$.

a. Prove the statement: for all integers a, b, n, and k, if $a \equiv_k b$, then $an \equiv_k bn$.

b. Prove the statement: there exists an integer k where, for all integers a and b, $a \equiv_k b$. (Hint: To prove an existentially quantified statement, first you'll need to come up with a value that works.)

This next problem needs some new notation, which won't come up elsewhere in CS 103. For two integers m and n, we say that $m \mid n$ if there exists an integer k where $n = km$. (Note: this is the same symbol as in set-builder notation, but it means something different here.)

c. Prove the statement: For all integers x, y, and z, if $x \mid y$ and $y \mid z$, then $x \mid z$.

2.5 Proof by Contradiction: Balls and Bins

Suppose that you have 11 balls to place into 5 different bins.

a. Consider this statement: there is a way to place the balls into the bins so that at least one bin contains at least 3 balls. Is this an existential or universal statement?

b. Prove the above statement.

c. Prove by contradiction that no matter how the balls are placed into the bins, there is at least one bin containing at least 3 balls. (Hint: How does this statement change when you take the negation?)

This is a sneak peek of a powerful mathematical principle we'll be discussing in week 4!