1. Set Theory

1.1 Set Theory Symbols

In our first lecture, we introduced a bunch of symbols in the context of set theory. As a reference, here's a list of all of the symbols we encountered:

$$\begin{array}{cccc} \in & \notin & \subseteq & \not\subseteq \\ \varnothing & \mathbb{N} & \mathbb{Z} & \mathbb{R} \\ \cup & \cap & \setminus & \Delta \\ \wp(\cdot) & |\cdot| & \aleph_0 & \end{array}$$

(Assume $\cdot$ is where the name of a set goes.)

What do each of these symbols mean? Give an example of how each might be used.

The first row of symbols are relationships between sets' elements. A statement involving one of these symbols is either true or false.

- The $\in$ symbol means **"element-of"**. It means that the object on the left side belongs to the set on the right side. For example, $1 \in \{1, 2, 3\}$.
- The $\notin$ symbol means **"not-element-of"**. It means that the object on the left side *doesn't* belong to the set on the right side. For example, $0 \notin \{1, 2, 3\}$.
- The $\subseteq$ symbol means **"subset-of"**. It means that all of the elements of the set on the left are also elements of the set on the right. For example, $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$.
- The $\not\subseteq$ symbol means **"not-subset-of"**. It means that some element(s) of the set on the left *doesn't* belong to the set on the right. For example, $\{1, 2, 3, 4\} \not\subseteq \{1, 2, 3\}$.

The second row of symbols all refer to particular sets:

- The $\varnothing$ symbol refers to **the empty set**, or $\{\}$, which doesn't have any elements. We could say that $1 \notin \varnothing$.
- $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$ refer to the set of all **natural numbers**, the set of all **integers**, and the set of all **real numbers**, respectively. (Why Z? It comes from the German word *Zahlen*, which means "numbers".)

The third row of symbols are set operations. Sets go on the left and right side of these symbols, and a statement involving one of these symbols will produce a set.

- The $\cup$ symbol refers to **set union**, producing a set that contains all the elements of both sets. For example, $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$.
- The $\cap$ symbol refers to **set intersection**, producing a set that contains only the elements that are in both sets. For example, $\{1, 2, 3\} \cup \{3, 4, 5\} = \{3\}$.

- The $\setminus$ symbol refers to **set difference**, producing a set with the elements from the left set that aren't in the right set. You can also write set difference with the regular $-$ sign. For example, $\{1, 2, 3\} \setminus \{3, 4, 5\} = \{1, 2\}$.
- The Δ symbol refers to **set symmetric difference**, producing a set with all the elements from either set that aren't in the other set. You can also think of it as the set union minus the set intersection. For example, $\{1, 2, 3\}\Delta\{3, 4, 5\} = \{1, 2, 4, 5\}$.

The last row are the rest of the set-related symbols introduced in the first lecture.

- The $\wp$ symbol refers to the **power set operation**, which takes in a set and produces the set of all its subsets. For example, $\wp(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.
- The vertical bars refer to **set cardinality**, or the number of elements in a set. For example, $|\{0, 1, 2\}| = 3$.
- The $\aleph_0$ symbol is **"aleph-0"**, which is a special number: the cardinality of the set $\mathbb{N}$. We can write that as $|\mathbb{N}| = \aleph_0$

1.2 Set Theory in Real Life: Pets and Sets

This problem practices using set operations and set-builder notation to express certain ideas. Say A is the set of all animals, B is the set of all brown things, C is the set of all cats, and D is the set of all dogs.

a. Write mathematical statements equivalent to these statements by using some of the sets A, B, C, D and some of the set theory symbols we've learned about:

(1) "All cats are animals."

> We could write this statement as "$C \subseteq A$": everything that is a cat also is an animal.

(2) "The number of brown cats is the same as the number of brown dogs."

> We could write this statement as "$|B \cap C| = |B \cap D|$". To express the set of brown cats, we can take the intersection of B and C, which will only contain elements that are both brown and cats. Then, the cardinality of this set is the number of brown cats. Similarly, the set of brown dogs is the intersection of the set of brown things and the set of dogs.

(3) "There isn't anything that is both a cat and a dog."

> One way to write this statement as "$C \cap D = \varnothing$": the set of things that are both cats and dogs is empty.

b. Use set-builder notation and some of the sets A, B, C, D to express these sets:

(1) The set of cute dogs

> We could write this set as $\{d \in D \mid d \text{ is cute}\}$. Two things to note about this answer: the variable name d could be anything as long as it's the same on both sides, and it's fine to write normal words on the right-hand side of the line in set-builder notation.

(2) The set containing everything that isn't a cat or a dog

> We could write this set as $\{x \mid x \notin C \cup D\}$ or $\{x \mid x \notin C \text{ and } x \notin D\}$. The set $C \cup D$ is the set of all cats and dogs, so if something is not in that set, it is not a cat or a dog. Again, the variable name x could be anything as long as it's the same on both sides.

1.3 Set-Builder Notation

Consider the following set:

$$S = \{n \in \mathbb{N} \mid n \text{ is odd}\}$$

Here's a mathematical argument. Is it correct? Why or why not?

> Let's pick the number $n = 137$. We know that n is odd, because 137 doesn't cleanly divide by two. We also know that n is a natural number, because it's a whole number and isn't negative. Since n is odd and $n \in \mathbb{N}$, $S = \{137\}$.

Nope, this argument is incorrect. The argument is correct that if we pick $n = 137$, then $n \in \mathbb{N}$ and n is odd, but that doesn't mean that $S = \{137\}$. Set-builder notation produces a set containing *every* element that satisfies the given conditions. For example, the number 103 is also odd and also a natural number, and so $103 \in S$. However, the statement $S = \{137\}$ means "the set S contains 137 and nothing else."

1.4 Exploring Sets

You'll probably need to play around with these properties a bit before you find something that works, so try things out and see what you come up with!

a. Find sets A and B where...

(1) $A \notin B$, but $A \subseteq B$.

> There are many answers here. One answer would be $A = \mathbb{N}$ and $B = \mathbb{Z}$. The set $\mathbb{N}$ is not itself an element of $\mathbb{Z}$, since $\mathbb{N}$ is a set and $\mathbb{Z}$ doesn't contain any sets. However, $\mathbb{N} \subseteq \mathbb{Z}$ because every element of $\mathbb{N}$ is also an element of $\mathbb{Z}$.

(2) $A \in B$, but $A \not\subseteq B$.

> One answer would be $A = \{1\}$ and $B = \{\{1\}\}$. Notice that $A \in B$ because the set $\{1\}$ itself is an element of B. However, A is not a subset of B: to see why, consider that A contains 1, but B does not, because it only contains the set containing 1.

(3) $A \in B$ and $A \subseteq B$.

> One answer would be $A = \varnothing$ and $B =$ any set containing the empty set, such as $\{\varnothing\}$. The empty set is a subset of all sets.

b. Find a set A where...

(1) $A \in \wp(A)$.

> Any set works! Since any set is a subset of itself, every set is contained within its power set.

(2) $A \subseteq \wp(A)$.

> This one's trickier! One simple answer is $\varnothing$, since it is a subset of all sets. Another answer would be $\{\varnothing\}$: since $\varnothing$ is a subset of all sets, it will be an element within $\wp(A)$. It's a good test to understand why these two answers work and differ from each other.

2. Mathematical Proofs

2.1 Properties of Odd and Even Numbers

a. Prove this statement: for any integer n, if n is even, then $n - 1$ is odd.

Proof: Pick an arbitrary even integer n. We'll show that $n - 1$ is odd.

Since n is even, there is an integer k such that $n = 2k$. Then we see that $n - 1 = 2k - 1 = 2(k - 1) + 1$. There is an integer m, namely $k - 1$, where $n - 1 = 2m + 1$, so $n - 1$ is odd, as required. ■

b. What is the smallest even natural number?

0. 0 is even, since there is an integer (namely 0) where $2 \cdot 0 = 0$. 0 is a natural number (in this class).

2.2 Proof by Contradiction: Pythagorean Triples

As a refresher, a Pythagorean triple is an ordered trio of positive natural numbers (a, b, c) where $a^2 + b^2 = c^2$.

Prove this theorem by **contradiction**: For any Pythagorean triple (a, b, c), if a^2 is even, then $c \neq b + 1$. (Hint: How does an implication change when you take the negation?)

How to approach this proof: Taking the negation of this statement, we need to use the pattern from lecture where taking the negation of a universally quantified statement with an implication inside it becomes an existentially quantified statement: "there exists a Pythagorean triple (a, b, c) where a^2 is even and $c = b + 1$."

We can actually use the same insights as the proof by contrapositive here, where we ended up concluding that a^2 is odd instead of being even.

Theorem: For all Pythagorean triples (a, b, c), if a^2 is even, then $c \neq b + 1$.

Proof: Assume for the sake of contradiction that there is a Pythagorean triple (a, b, c) where a^2 is even and $c = b + 1$. Because (a, b, c) is a Pythagorean triple, we know that

$$a^2 + b^2 = c^2.$$

Substituting $c = b + 1$ into this equation, we see that

$$a^2 + b^2 = (b + 1)^2 = b^2 + 2b + 1.$$

We can simplify this equation to find that $a^2 = 2b + 1$, meaning that a^2 is odd. However, this is impossible: we assumed that a was even, and no number can be both odd and even. We have reached a contradiction, so our assumption must have been wrong. Therefore, for all Pythagorean triples (a, b, c), if a^2 is even, then $c \neq b + 1$. ■

2.3 Proof by Contrapositive: Multiples of Four

In lecture, we proved that if n is an integer, then n is even if and only if n^2 is even. This question explores some other properties about the relationship between n and n^2, giving you a chance to practice with proofs and proof techniques.

An integer n is called a **multiple of four** if n is equal to $4k$ for some integer k. Consider this statement:

For any integer n, if n^2 is not a multiple of four, then n is odd. (★)

a. Proof by contrapositive

(1) What is the contrapositive of the implication in statement (★)?

> The antecedent is "n^2 is not a multiple of four" and the consequent is "n is odd". The negation of the antecedent is "n^2 is a multiple of four" and the negation of the consequent is "n is even." This means the contrapositive is "if n is even, then n^2 is a multiple of four".

(2) Prove statement (★) using a proof by contrapositive.

> **Proof:** Let n be an arbitrary integer. We will prove the contrapositive of the statement, namely that if n is even, then n^2 is a multiple of four. Since n is even, there must be an integer k such that $n = 2k$. Therefore, we see that $n^2 = (2k)^2 = 4k^2$. This means that there is an integer m, namely, k^2, such that $n^2 = 4m$. Thus, n^2 is a multiple of four, as required.

b. Prove that if n is odd, there exists an integer k where $n^2 = 4k + 1$.

In combination with the contrapositive of statement (★), we can see that every perfect square is either a multiple of four or 1 greater than a multiple of four. Pretty cool!

> **Proof:** Let n be an arbitrary integer. We will prove the contrapositive of the statement, namely that if n is even, then n^2 is a multiple of four. Since n is even, there must be an integer k such that $n = 2k$. Therefore, we see that $n^2 = (2k)^2 = 4k^2$. This means that there is an integer m, namely, k^2, such that $n^2 = 4m$. Thus, n^2 is a multiple of four, as required.

2.4 Proving Mixed Statements

I recommend trying these problems after the problem set 1 question on modular congruence!

Here's a refresher on modular congruence: For an integer k, we say that $a \equiv_k b$ when there exists an integer q such that $a = b + kq$.

a. Prove the statement: for all integers a, b, n, and k, if $a \equiv_k b$, then $an \equiv_k bn$.

> **Proof:** Let a, b, n, and k be arbitrary integers where $a \equiv_k b$. We need to show that $an \equiv_k bn$. To do so, we will show that there is an integer q where $an = bn + kq$.
>
> Since we know $a \equiv_k b$, we know there is an integer r where $a = b + kr$. Then, we see that
> $$\begin{aligned} an &= (b + kr)n \\ &= bn + krn. \end{aligned}$$
>
> Therefore, there is an integer q (namely, kr) such that $an = bn + kq$. This means that $an \equiv_k bn$, which is what we needed to show. ■

b. Prove the statement: there exists an integer k where, for all integers a and b, $a \equiv_k b$. (Hint: To prove an existentially quantified statement, first you'll need to come up with a value that works.)

> **Proof:** We will show that for all integers a and b, $a \equiv_1 b$. To do so, pick arbitrary integers a and b; we'll show that $a \equiv_1 b$. Notice that $a = b + (a - b)$. Therefore, there is an integer q (namely, $a - b$) where $a = b + 1 \cdot q$, meaning that $a \equiv_1 b$, as required.
>
> We've shown that there is a number k, namely 1, where for all integers a and b, $a \equiv_k b$, as required. ■
>
> (Note: -1 would also work here!)

This next problem needs some new notation, which won't come up elsewhere in CS 103. For two integers m and n, we say that $m \mid n$ if there exists an integer k where $n = km$. (Note: this is the same symbol as in set-builder notation, but it means something different here.)

c. Prove the statement: For all integers x, y, and z, if $x \mid y$ and $y \mid z$, then $x \mid z$.

> **Proof:** Pick any integers x, y, and z where $x \mid y$ and $y \mid z$. We'll show that $x \mid z$. To do so, we'll show that there is an integer k where $z = kx$.
>
> Since we know that $x \mid y$, we know there is an integer n_1 where $y = n_1 x$. And since we know that $y \mid z$, we know there is an integer n_2 where $z = n_2 y$. Then, we can see that $z = n_2 y = n_2(n_1 x)$. This means there is an integer k, namely $n_2 n_1$, where $z = kx$. This

means that $x \mid z$, which is what we wanted to show. ■

2.5 Proof by Contradiction: Balls and Bins

Suppose that you have 11 balls to place into 5 different bins.

a. Consider this statement: there is a way to place the balls into the bins so that at least one bin contains at least 3 balls. Is this an existential or universal statement?

> This is an existential statement: "there is" something.

b. Prove the above statement.

> One way to put the balls into the bins would be to just put all 11 balls into the same bin. Then, that bin would have at least 3 balls.

c. Prove by contradiction that no matter how the balls are placed into the bins, there is at least one bin containing at least 3 balls. (Hint: How does this statement change when you take the negation?)

> **How to approach this proof:** Whenever we're given a statement saying that something is true no matter what, it can be a good idea to try out different examples. Intuitively, it seems like even if we try to spread out the balls as much as possible, there will always be some bin with 3 or more balls.
>
> Since we've been asked to write this proof by contradiction, let's see what the negation of this statement would be. We can write this statement in a different way: "For any way to place the balls into the bins, there is at least one bin containing at least 3 balls." This is a universal statement wrapped around an existential statement. So the negation of this statement is: "There is a way to place the balls into the bins so that every bin contains less than 3 balls."
>
> We can formalize the above intuition by exploring what would happen with this assumption. If every bin had 2 or fewer balls, how many would that use up? Since there are 5 bins, we'd only be able to put in 10 balls total, so we would have one leftover that would make one of the bins go above 2 balls.
>
> Note that this proof doesn't incorporate any definitions or formal mathematics, but we can still negate the statement and use the techniques of a proof by contradiction!

> **Theorem:** Given 11 balls to place into 5 bins, no matter how the balls are placed into the bins, there is at least one bin containing at least 3 balls.
>
> **Proof:** Assume for the sake of contradiction that there is a way to place the balls into the bins so that every bin contains no more than 2 balls. Since there are 5 bins, the total number of balls placed into bins would be no more than 10. However, this contradicts

the fact that we have 11 total balls, so our assumption must have been wrong and there must be at least one bin containing at least 3 balls no matter how we place the balls into the bins. ■

This is a sneak peek of a powerful mathematical principle we'll be discussing in week 4!