1. Propositional Logic Review

Expression	English Translation	Negation of Expression
$A \land B$	A and B	2 possibilities: $\neg A \lor \neg B$, $A \to \neg B$
$A \lor B$	A or B	$\neg A \land \neg B$
$\neg A$	Not A	A
$A \to B$	A implies B	$A \land \neg B$
$A \leftrightarrow B$	A if and only if B	2 possibilities: $A \leftrightarrow \neg B$, $\neg A \leftrightarrow B$
$\top$	True	$\bot$
$\bot$	False	$\top$

2. Negating Propositional Statements

We'll demonstrate "starting from the outside" when taking negations in class. Take the negation of the following statements:

a. $A \to \neg B$

> The outermost connective here is $\to$. The negation of the implication is $A \land \neg(\neg B)$, which simplifies to $A \land B$. This shows why this expression is one of the negations in the first row of the table above.

b. $(C \to D) \land (D \to C)$

> The outermost connective here is $\land$. That means the negation is $\neg(C \to D) \lor \neg(D \to C)$. Then, we can negate each of the implications to find that the negation is $(C \land \neg D) \lor (D \land \neg C)$.
>
> We can also observe that this original statement is equivalent to $C \leftrightarrow D$, so we can also substitute either of the negations from the table above.

c. $\neg Q \to \neg P$

> The outermost connective here is $\to$. The negation of the implication is $\neg Q \land \neg(\neg P)$. Simplifying the second term, we see that this becomes $\neg Q \land P$.
>
> We can also observe that this original statement is equivalent to $C \leftrightarrow D$, so we can also substitute either of the negations from the table above.

d. $(\neg X \land Y) \lor Z$

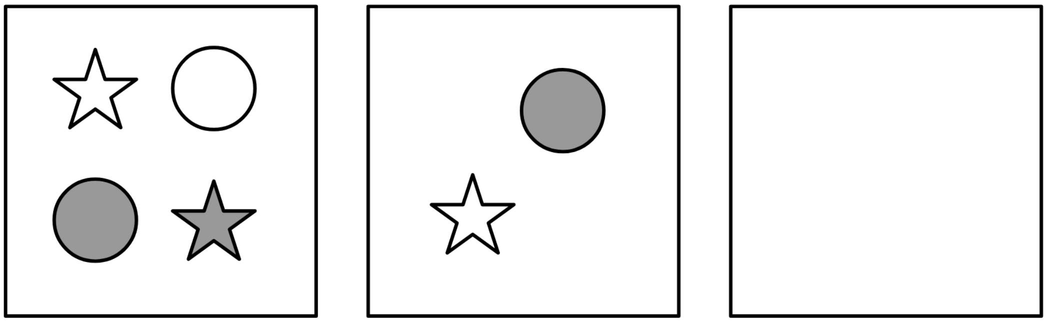
> The outermost connective here is $\lor$, so the negation is:
>
> $\neg(\neg X \land Y) \land \neg Z$
>
> To simplify the first part of the expression, we take the negation of an "and" statement:
>
> $((\neg(\neg X)) \lor \neg Y) \land \neg Z$
>
> $(X \lor \neg Y) \land \neg Z$
>
> Bonus: Notice that this result takes the form of "A and not B": the negation of an implication! That means the original expression is equivalent to the implication "A implies B", or $(X \lor \neg Y) \to Z$.

3. First-Order Logic I Review

Expression	English Translation
$\forall x.(P(x))$	Everything is a P.
$\exists x.(P(x))$	Something is a P.
$\forall x.(A(x) \to B(x))$	All A's are B's.
$\exists x.(A(x) \land B(x))$	Some A is a B.

4. Evaluating First-Order Logic Statements

Let's say the predicate $Gray(x)$ is true if x is gray, $White(x)$ is true if x is white, $Star(x)$ is true if x is a star, and $Circle(x)$ is true if x is a circle. Consider the following three worlds:



World 1 World 2 World 3

For each of the following first-order logic statements, say whether it's true in each of the worlds.

Expression	World 1	World 2	World 3
$\exists x.\big(Circle(x)\big)$	$\top$	$\top$	$\bot$
$\forall x.\big(Circle(x)\big)$ In worlds 1 and 2, the stars are counterexamples.	$\bot$	$\bot$	$\top$
$\forall x.\big(White(x) \to Star(x)\big)$ The white circle in world 1 is a counterexample.	$\bot$	$\top$	$\top$
$\forall x.\big(Star(x) \to White(x)\big)$ The gray star in world 1 is a counterexample.	$\bot$	$\top$	$\top$
$\exists x.\big(Gray(x) \land Star(x)\big)$ There is only a gray star in world 1.	$\top$	$\bot$	$\bot$
$\exists x.\big(Gray(x) \lor Star(x)\big)$ Either choice of shape works in world 2.	$\top$	$\top$	$\bot$
$\forall x.\big(White(x) \leftrightarrow Circle(x)\big)$ The gray circles and white stars are counterexamples.	$\bot$	$\bot$	$\top$

5. Translating English to First-Order Logic

Translate each statement into first-order logic, given these predicates: $HasHat(x)$ says that x is wearing a hat and $Dog(x)$ says that x is a dog.

a. There is a dog.

> This takes the form of one of the basic forms:
>
> $\exists d. \big(Dog(d)\big)$

b. All dogs wear hats.

> We can rephrase this to "all dogs are hat-wearers", which is one of our basic first-order logic forms:
>
> $\forall d. \big(\text{d is a dog} \to \text{d is wearing a hat}\big)$
>
> $\forall d. \big(Dog(d) \to HasHat(d)\big)$

c. There is a dog with a hat.

> We can rephrase this to "some dogs are hat-wearers", which is one of our basic first-order logic forms:
>
> $\exists d. \big(\text{d is a dog} \land \text{d is wearing a hat}\big)$
>
> $\exists d. \big(Dog(d) \land HasHat(d)\big)$

d. Some dogs don't wear hats.

> We can rephrase this to "some dogs are not hat-wearers", which is one of our basic first-order logic forms:
>
> $\exists d. \big(\text{d is a dog} \land \text{d is not wearing a hat}\big)$
>
> $\exists d. \big(Dog(d) \land \neg HasHat(d)\big)$

e. Some dogs wear hats, but not all dogs wear hats.

> The main, outermost connective in this statement is the "but", which can be translated into first-order logic as "and". We already know how to say "some dogs wear hats" from part (c), and we know how to say "some dogs don't wear hats" from part (d). If we combine them with $\land$, we get our answer:

$$\exists d_1.\ \left(Dog(d_1) \land HasHat(d_1)\right) \land \exists d_2.\ \left(Dog(d_2) \land \neg HasHat(d_2)\right)$$

(Note: We don't need to use separate variable names here, since d_1 and d_2 are in different scopes. But I think it helps for readability.)

6. First-Order Logic II Review

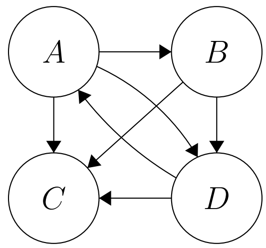
The table from First-Order Logic I Review is reproduced for your quick reference.

Expression	English Translation	Negation of Expression
$\forall x.(P(x))$	Everything is a P.	$\exists x.(\neg P(x))$
$\exists x.(P(x))$	Something is a P.	$\forall x.(\neg P(x))$
$\forall x.(\neg P(x))$	Nothing is a P.	$\exists x.(P(x))$
$\exists x.(\neg P(x))$	Something isn't a P.	$\forall x.(P(x))$
$\forall x.(A(x) \to B(x))$	All A's are B's.	$\exists x.(A(x) \land \neg B(x))$
$\exists x.(A(x) \land B(x))$	Some A is a B.	$\forall x.(A(x) \to \neg B(x))$
$\forall x.(A(x) \to \neg B(x))$	No A is a B.	$\exists x.(A(x) \land B(x))$
$\exists x.(A(x) \land \neg B(x))$	Some A isn't a B.	$\forall x.(A(x) \to B(x))$

7. Evaluating Nested First-Order Logic Statements

For each statement, translate it into English, then decide whether it's true or false.

Interpersonal Dynamics: This diagram represents a set P of people named A, B, C and D. If there's an arrow from a person x to a person y, then person x loves person y. We'll denote this by writing $Loves(x, y)$.



a. $\exists x \in P.\ Loves(x, x)$

> This means "there is some person x who loves themself."
>
> This is not true! No one in this diagram has an arrow toward themself.

b. $\forall y \in P.\ \exists x \in P.\ Loves(x, y)$

> This means "for every person y, we can pick someone who loves y."
>
> This is true: A is loved by D, B is loved by A, C is loved by everyone, and D is loved by A.

c. $\exists x \in P.\ \forall y \in P.\ Loves(x, y)$

> This means "there is some person x who loves every person in P."
>
> This is not true: since no one in this diagram loves themself, no matter who you pick for x, you can pick that same person for y and the statement will not be true.
>
> The key difference between this part and the previous part is that in this part, you need to find one single person x, but in the previous part, x could change depending on what you picked for y.

d. $\exists x \in P.\ \forall y \in P.\ (x \neq y \to Loves(x, y))$

> This means "there is some person x who loves every *other* person y in P."
>
> This is true! We can pick A as person x, and A loves all the other people in the diagram (B, C, and D).

e. $\forall x \in P.\ \forall y \in P.\ (x \neq y \to (Loves(x, y) \lor Loves(y, x)))$

This means "for every person x and every other person y, either x loves y or y loves x (or both)."

This is true! No matter which two people we pick, there is a relationship between them.

f. $\forall x \in P.\ \forall y \in P.\ (x \neq y \to (Loves(x, y) \leftrightarrow \neg Loves(y, x)))$

This means "for every person x and every other person y, x loves y if and only if y doesn't love x."

This is not true: pick x to be A and y to be D. The antecedent of the statement is true, since $A \neq D$. Then, consider the biconditional in the consequent. The left side of the biconditional is true since A loves D. The right side of the biconditional is false since D loves A, making the entire biconditional statement false. This makes the entire statement "true implies false", which is false.

8. Translating English into First-Order Logic II

Translate each statement into first-order logic, given these predicates: $Dog(x)$ says that x is a dog, $Robot(x)$ says that x is a robot, and $Loves(x, y)$ says that x loves y.

a. Some robot loves exactly one dog. (You can express "there is exactly one thing with a certain property" by saying "there is something with that property, and if something else has that property, then they're the same thing.")

> Since this is stating something that is true of only one/some robots, we can start by applying the "some P's are Q's" form:
>
> $\exists r.(Robot(r) \land \text{r loves exactly one dog})$
>
> Following the hint – in this case, the property we care about is being the dog that is loved by the robot, which we can express as follows:
>
> $\exists r.(Robot(r) \land \text{there is a dog that r loves } \land \text{ there is no other dog that r loves})$
>
> Taking one expression at a time, we have:
>
> $\exists r.(Robot(r) \land \exists d.(Dog(d) \land Loves(r, d) \land \text{there is no other dog that r loves}))$
>
> And our final answer is:
>
> $$\exists r.\Big(Robot(r) \land \exists d.\big(Dog(d) \land Loves(r, d) \land \forall x.(Dog(x) \land Loves(r, x) \to d = x)\big)\Big)$$
>
> Notice that we could also express the final statement via the contrapositive of what is written, i.e. "anything that isn't d is either not a dog or not loved by r". I find this slightly harder to understand, but it's still correct:
>
> $$\exists r.\Big(Robot(r) \land \exists d.\big(Dog(d) \land Loves(r, d) \land \forall x.(d \neq x \to \neg Dog(x) \lor \neg Loves(r, x))\big)\Big)$$

b. There are at least two dogs. (You can talk about multiple objects by nesting quantifiers and, if necessary, checking that the two objects you are looking at are not the same. See the Guide to Logic Translation for more on this.)

> We can translate this statement as "there exist two dogs", or rather: "there is a dog, and there is a dog that is different from the first dog."
>
> We translate the first part as follows:
>
> $\exists x.(Dog(x) \land \text{there is a dog that is different from x})$
>
> Again, the second part can be translated as an existentially quantified statement:
>
> $\exists x.\big(Dog(x) \land \exists y.(Dog(y) \land y \neq x)\big)$

9. Negating Statements in First-Order Logic

Negate each of these first-order logic formulas below. The only negations your final formula should have are direct negations of predicates. For example, the negation of the formula $\forall x.\ (P(x) \to \exists y.\ (Q(x) \land R(y)))$ could be found by pushing the negation from the outside inward as follows:

$$\neg(\forall x.\ (P(x) \to \exists y.\ (Q(x) \land R(y))))$$
$$\exists x.\ \neg(P(x) \to \exists y.\ (Q(x) \land R(y)))$$
$$\exists x.\ (P(x) \land \neg(\exists y.\ (Q(x) \land R(y))))$$
$$\exists x.\ (P(x) \land \forall y.\ \neg(Q(x) \land R(y)))$$
$$\exists x.\ (P(x) \land \forall y.\ (Q(x) \to \neg R(y)))$$

Show every step of the process of pushing the negation into the formula.

a. $\exists k.\ \big(Coder(k) \land Athlete(k) \land Painter(k)\big)$

(Hint: Add parentheses to make the inside statement look more like a basic form.)

Here's one way we could add parentheses:

$$\neg\Big(\exists k.\ \big(Coder(k) \land (Athlete(k) \land Painter(k))\big)\Big)$$
$$\forall k.\ \neg\big(Coder(k) \land (Athlete(k) \land Painter(k))\big)$$
$$\forall k.\ \big(Coder(k) \to \neg(Athlete(k) \land Painter(k))\big)$$
$$\forall k.\ \big(Coder(k) \to \neg Athlete(k) \lor \neg Painter(k)\big)$$

Note that there are many equivalent statements to the inner statement. Any of these statements would also work:

$$\forall k.\ \big(Coder(k) \land Athlete(k) \to \neg Painter(k)\big)$$
$$\forall k.\ \big(\neg Coder(k) \lor \neg Athlete(k) \lor \neg Painter(k)\big)$$
$$\forall k.\ \big(Coder(k) \to (Athlete(k) \to \neg Painter(k))\big)$$

b. $\forall t.\ \big(Leafy(t) \land Thorny(t) \to Plant(t)\big)$

Note that $\land$ takes precedence over $\to$, so the inside statement is equivalent to $(Leafy(t) \land Thorny(t)) \to Plant(t)$. Here's one approach:

$$\neg\Big(\forall t.\ \big(Leafy(t) \land Thorny(t) \to Plant(t)\big)\Big)$$
$$\exists t.\ \neg\big(Leafy(t) \land Thorny(t) \to Plant(t)\big)$$

$$\exists t.\ \big(Leafy(t) \land Thorny(t) \land \neg Plant(t)\big)$$

c. $\exists r.\ \big(Silly(r) \leftrightarrow \neg Serious(r)\big)$

There are two negations for $\leftrightarrow$ in the table. The one that we apply here leads to having no negations in the final product:

$$\neg\Big(\exists r.\ \big(Silly(r) \leftrightarrow \neg Serious(r)\big)\Big)$$

$$\forall t.\ \neg\big(Silly(r) \leftrightarrow \neg Serious(r)\big)$$

$$\forall t.\ \big(Silly(r) \leftrightarrow Serious(r)\big)$$

d. $\exists u.\ \Big(Unicorn(u)\Big) \to \exists h.\ \Big(Horse(h) \land Magical(h)\Big)$

Here, note that the entire expression has the form $p \to q$, so we start by applying the negation listed in the propositional logic table.

$$\neg\Bigg(\exists u.\ \Big(Unicorn(u)\Big) \to \exists h.\ \Big(Horse(h) \land Magical(h)\Big)\Bigg)$$

$$\exists u.\ \Big(Unicorn(u)\Big) \land \neg\Bigg(\exists h.\ \Big(Horse(h) \land Magical(h)\Big)\Bigg)$$

Now the expression we're negating takes exactly one of the four basic forms, leading to our final answer:

$$\exists u.\ \Big(Unicorn(u)\Big) \land \forall h.\ \Big(Horse(h) \to \neg Magical(h)\Big)$$

e. $\forall x.\ \Big(Person(x) \to \exists y.\ \exists z.\big(CanJuggle(x,y) \land \neg CanJuggle(x,z)\big)\Big)$

Here's one approach:

$$\neg\Bigg(\forall x.\ \Big(Person(x) \to \exists y.\ \exists z.\big(CanJuggle(x,y) \land \neg CanJuggle(x,z)\big)\Big)\Bigg)$$

$$\exists x.\ \neg\Big(Person(x) \to \exists y.\ \exists z.\big(CanJuggle(x,y) \land \neg CanJuggle(x,z)\big)\Big)$$

$$\exists x.\ \Big(Person(x) \land \neg\big(\exists y.\ \exists z.\big(CanJuggle(x,y) \land \neg CanJuggle(x,z)\big)\big)\Big)$$

$$\exists x.\ \Big(Person(x) \land \forall y.\ \forall z. \neg\big(CanJuggle(x,y) \land \neg CanJuggle(x,z)\big)\Big)$$

$$\exists x.\ \Big(Person(x) \land \forall y.\ \forall z.\big(CanJuggle(x,y) \to CanJuggle(x,z)\big)\Big)$$

In this last step, again, I chose the negation that leads to the least "not"s, but it would also be valid to write $\neg CanJuggle(x, y) \lor CanJuggle(x, z)$.

f. $\forall p. \Big(Person(p) \to \big(\exists q. \big(Person(q) \land TallerThan(p, q)\big)\big) \lor \big(\exists q. \big(Person(q) \land TallerThan(q, p)\big)\big)\Big)$

I'm going to define some new "variables": $\bigstar$ represents the expression $\exists q.(Person(q) \land TallerThan(p, q))$ and $\diamondsuit$ represents the expression $\exists q.(Person(q) \land TallerThan(q, p))$. Then, we can simplify the negation of the original expression without as many nested parentheses potentially confusing us:

$$\neg(\forall p.(Person(p) \to \bigstar \lor \diamondsuit))$$
$$\exists p.\neg(Person(p) \to \bigstar \lor \diamondsuit)$$
$$\exists p.(Person(p) \land \neg(\bigstar \lor \diamondsuit))$$
$$\exists p.(Person(p) \land \neg\bigstar \land \neg\diamondsuit)$$

Now we just need to figure out what $\neg\bigstar$ and $\neg\diamondsuit$ are! Here's the former:

$$\neg(\exists q.(Person(q) \land TallerThan(p, q)))$$
$$\forall q.\neg(Person(q) \land TallerThan(p, q)))$$
$$\forall q.(Person(q) \to \neg TallerThan(p, q))$$

The latter is worked out in the same way, coming to $\forall q.(Person(q) \to \neg TallerThan(q, p))$.

Putting everything together, our final answer is:

$$\exists p. \Big(Person(p) \land \forall q. \big(Person(q) \to \neg TallerThan(p, q)\big) \land \forall q. \big(Person(q) \to \neg TallerThan(q, p)\big)\Big)$$

Key tips for negations:

- Go slowly, one step at a time.
- Understand the parentheses in the formula before you start the process.
- If you have a complicated expression, replace it with a symbol and negate it later.