1. Implications Are Weird

The "implies" connective $\to$ is one of the stranger connectives. Let P, Q, and R be propositions. Below are a series of statements. For each statement, explain whether it is always true, always false, or depends on the values of the propositions involved. (Hint: draw truth tables!)

a. $P \to \neg P$

b. $(P \to Q) \lor (Q \to P)$

c. $(P \to Q) \lor (Q \to R)$

2. Designing Propositional Formulas

Here are some English descriptions of relationships among propositional variables. For each description, write a propositional formula that means the same thing. Then, briefly explain why your formula works. Try to see if you can come up with the simplest formula possible.

(If you find yourself writing out something extremely long, try rethinking your approach. All of these problems have formulas that can be written out in a single line.)

a. For the variables A, B, and C: Exactly one of A, B, and C is true. Our solution has 8 connectives, not including $\neg$.

b. For the variables A, B, C, and D: If any variable is true, then all the variables that follow it alphabetically are also true. (Hint: Try this out with just two variables first, then three variables.) Our solution has 5 connectives.

3. More Translating First-Order Logic to English

Translate these first-order logic statements to English, then say whether or not they are true.

a. **Numbers:**

(1) $\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ n < m$

(2) $\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ n < m$

(3) $\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ (n < m \land \exists p \in \mathbb{N}.\ (n < p \land p < m))$

(4) $\forall n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ (n < m \to \exists p \in \mathbb{N}.\ (n < p \land p < m))$

(5) $\exists n \in \mathbb{Z}.\ (2n > n \leftrightarrow 2n < n)$

4. More Translating English to First-Order Logic

Given these predicates:

- $Old(x)$, which says x is old,
- $Dog(d)$, which says x is a dog,
- $Trick(t)$, which says t is a trick,
- $CanTeach(x, y)$, which says that you can teach y to x

a. Write a first-order logic expression that means "Everything that you can teach to some dog has to be a trick."

b. Write a first-order logic expression that means "Only dogs can learn tricks." (Interpret this as meaning "If there are no dogs, nothing can be taught tricks.")

c. Write a first-order logic expression that means "There's exactly one old trick."

d. Write a first-order logic expression that means "Every dog can be taught tricks, and some dogs can even be taught all tricks."

e. Write a first-order logic expression that means "you can't teach an old dog new tricks". (Assume you can say something is "new" by saying it is "not old".)

5. More Negations

Look up the first-order logic statement corresponding to each of the statements from the solutions for "Translating English to First-Order Logic", then take the negation.

6. First-Order Logic in Arguments: Epimenides's Meal Plan

Below are some flawed arguments about certain statements. Identify the flaws in each argument. (Hint: Write out each statement in first-order logic.)

a. **Situation:** Epimenides, who is a Cretan, says "all Cretans always lie."

   Incorrect Argument: We'll show that Epimenides's statement is a paradox (a statement that cannot be true or false). If Epimenides tells the truth, then all Cretans always lie. Since Epimenides is himself a Cretan, he must be lying, which is impossible because we assumed that Epimenides is telling the truth. This is a contradiction.

   If, on the other hand, Epimenides is lying, then his statement is false and all Cretans never lie. Since Epimenides himself is a Cretan, then he must be telling the truth, which is impossible because we know that he was lying. This is also a contradiction. Therefore, this statement cannot be true or false.

b. **Statement:** In every non-empty dining hall, there is someone in the dining hall who could truthfully say "if I am eating, then everyone in this dining hall is eating."

Incorrect Argument: We'll show that this statement is false. Since this statement makes a universal claim, we can disprove it by identifying a counterexample: a dining hall where there is someone who would be lying if they said "if I am eating, then everyone in this dining hall is eating." Consider a dining hall d where some person p in d is eating and some other person in d is not eating. p would be lying if they said "if I am eating, then everyone in d is eating", because the antecedent is true but the consequent is false. Then, there exists a dining hall that does not have the property in the statement, so the statement is false.