1. Reviewing Graph Definitions

Graphs come with a lot of specific terminology. Here are some questions to review the terms.

a. What does it mean for two nodes to be adjacent in a graph?

b. Can undirected graphs have self-loops? Can directed graphs? Why/why not?

c. What is a walk in a graph? How do you measure the length of a walk? What is a closed walk?

d. What is a path in a graph? How is it different from a walk? Can a path be a closed walk?

e. What is a cycle in a graph? How is it different from a path?

f. What does it mean for two nodes to be connected in a graph? (This is also sometimes called being reachable.) How is this different from being adjacent?

g. Is it possible for two nodes in a graph to be adjacent but not connected?

h. Is it possible for two nodes in a graph to be connected but not adjacent?

i. What does it mean for a graph to be connected?

j. What is a connected component in a graph?

k. How many connected components does each node in a graph belong to?

l. What is meant by the degree of a node in an undirected graph? What about the in-degree and the out-degree in a directed graph?

2. Applying Definitions on Graphs

Many proofs on graphs deal with properties expressed in first-order logic. This problem demonstrates some strategies you can use when interpreting new formal definitions.

Given a graph $G = (V, E)$, G is called **triangle-free** if the following property holds:

$$\forall u \in V.\ \forall v \in V.\ \forall w \in V.\ \big((\{u, v\} \in E \land \{v, w\} \in E) \to \{u, w\} \notin E\big)$$

Given a graph $G = (V, E)$ and a specific node $u \in V$, let the **neighborhood set** of u be the set $\{v \in V \mid (u, v) \in E\}$.

Finally, here's the formal definition of an **independent set** again: In a graph $G = (V, E)$, a set $I \subseteq V$ is an independent set if the following property holds:

$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E$$

We'll prove the following property of graphs: For any triangle-free graph $G = (V, E)$ and node $u \in V$, the neighborhood set of u is an independent set.

a. One strategy for approaching definitions is to start by *breaking down the structure of the definition.* We'll do this with the triangle-free definition.

  (1) In the definition of triangle-free, do the variables u, v, and w represent nodes or edges of the graph?

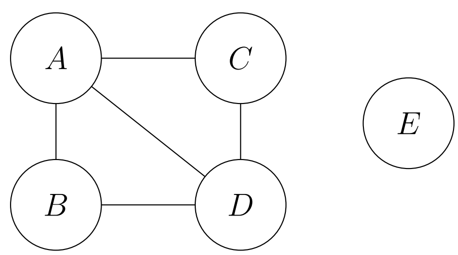
  (2) Is the definition universally or existentially quantified?

  (3) If you wanted to check if a graph *is* triangle-free, what would you have to do: find a specific counterexample, or check a property holds for all possible choices of u, v, w?

  (4) If you wanted to check if a graph is *not* triangle-free, what would you have to do: find a specific counterexample, or check a property holds for all possible choices of u, v, w? (Hint: Take the negation of the definition.)

  (5) Explain the triangle-free property in words.

b. Another strategy is to *try small examples to explore the definition.* We'll do this with the neighborhood set definition. Here's a small graph we can use to try this out.



  (1) Could a node ever be in its own neighborhood set? Why or why not?

  (2) What is the neighborhood set of C? How about D? And E?

  To check your answers, you should be able to point to each node that is in your neighborhood set and justify to yourself why it meets the criteria on the right-hand-side of the set-builder notation. You should also be able to point to every node that is *not* in your neighborhood set and justify to yourself why it doesn't meet those criteria.

  (3) Explain what a neighborhood set is in words.

c. Bonus: let's try to connect the definitions along the lines of the proof.

  (1) Which of the neighborhood sets you found in (b)(2) are independent sets?

  (2) The graph from part (b) is not triangle-free. What is the smallest number of edges you have to remove to make it triangle-free? (Note: We will not be talking about removing edges from a graph in the proof. This is just to get a look at an actually triangle-free graph.)

d. With a better intuitive understanding of the definitions, let's move on to *setting up the proof.*

Here's the theorem we're trying to prove again: For any triangle-free graph $G = (V, E)$ and node $u \in V$, the neighborhood set of u is an independent set.

(1) What should our assume and want-to-show be? How should the graph G and node u be picked: can they be picked arbitrarily by the reader, or should you give a specific example?

(2) Write out the definitions of the properties in your assume and want-to-show columns. (Since we have a specific node u mentioned in the problem, you'll probably want to change the variable names in the definitions.)

(3) We are assuming a universally quantified statement and trying to prove a universally quantified statement. Which variables should we introduce — corresponding to the variables in the statement we're assuming, or corresponding to the statement we're trying to show?

e. After setting up the proof, work it out using the two-column organizer and write your proof.

3. Applying the Pigeonhole Principle

a. Say there are 8,000 undergrads at Stanford. There are 366 possible birthdays. Fill in the blanks with the largest possible number guaranteed by the Pigeonhole Principle:

(1) There must be at least __________ undergrad(s) who have the same birthday as one another.

(2) There must be at least __________ undergrad(s) who were born on February 29.

b. Fill in the blank with the smallest possible number guaranteed by the Pigeonhole Principle:

(1) There is some day of the year with at most __________ undergrad(s) who have that birthday.

(2) There are 7 days of the week. If we have a group of __________ people, we are guaranteed that two of them were born on the same day of the week.

(3) There are 26 letters of the alphabet. If we have a group of __________ people, we are guaranteed that at least 5 of them have the same first initial.

c. Suppose you pick 11 numbers from the set $\{n \in \mathbb{N} \mid 1 \leq n \leq 20\}$. We'll prove that out of those 11 numbers, there must be at least one pair of numbers whose difference is exactly 10.

- First, try this exercise yourself: write down eleven numbers, and say which ones differ by exactly 10. Then, write down ten numbers without including any pairs of numbers that differ by exactly 10. (There are many ways to do this!)

  What happens when you try to add an eleventh number to your list of ten? Can you generalize this into a claim about having too many "pigeons" for your "holes"?