1. Reviewing Definitions from Lecture

a. How would you show that a set is not an independent set? (Negate the formal definition.)

> Find two nodes in the set that are adjacent.

b. How would you show that a set is not a vertex cover? (Negate the formal definition.)

> Find some edge in the graph where neither endpoint of the edge is in the set.

c. Explain the Theorem on Friends and Strangers in your own words.

> Here's the graph theory view: If we have a complete graph with 6 nodes, and we color every edge of the graph either one color or another color, then there has to be a triangle of nodes that are all connected with the same color.
>
> Here's the analogy view: If there's a party with 6 people, and they are all either friends or strangers with each other, then there have to be three people who are all friends, or three people who are all strangers.

2. Graphs, Sets, and Logic

a. Graphs are defined in terms of sets. Let's explore. Given a graph $G = (V, E)$ and a node $v \in V$, write these sets in set-builder notation:

(1) "the set containing all edges touching v"

> Here are two ways we could say this:
>
> $\{\{a, b\} \in E \mid v \in \{a, b\}\}$
>
> $\{\{a, b\} \in E \mid v = a \lor v = b\}$
>
> Notice that, as a set containing edges, this set is a subset of E. Also, v does not appear on the left-hand side of $\mid$ because the left-hand side is for only placeholder variables, while v has a specific value.

(2) "the set containing all nodes adjacent to v"

> $\{u \in V \mid \{u, v\} \in E\}$
>
> Notice that, as a set containing nodes, this set is a subset of V.

b. Using the predicate $IsPath(a, b, G)$, which says that there is a path between nodes a and b in the graph G, write these statements in first-order logic:

(1) "G has exactly one connected component"

If G has exactly one connected component, we know that every node is reachable from every other node.

$\forall u \in V. \Big(\forall v \in V. \big(u \neq v \rightarrow IsPath(u, v, G) \big) \Big)$

(2) "G has more than one connected component"

Notice that this is the negation of "G has exactly one connected component".

$\exists u \in V. \Big(\exists v \in V. \big(u \neq v \land \neg IsPath(u, v, G) \big) \Big)$

3. Applying Definitions on Graphs

Prove the following statement:

For any graph $G = (V, E)$: if for any node $v \in V$, the neighborhood set of v is an independent set, then G is triangle-free.

Proof: Let $G = (V, E)$ be an arbitrary graph where, for any node $v \in V$, the neighborhood set of v is an independent set. We'll show that G is triangle-free.

To do so, pick three nodes a, b, c from V where $\{a, b\} \in E$ and $\{a, c\} \in E$. We'll show that there is no edge $\{b, c\}$ in E. Let N be the neighborhood set of a, which we know is an independent set. Because we know that $\{a, b\} \in E$, we know that b is in N, and because we know that $\{a, c\} \in E$, we know that c is in N. Then, because N is an independent set, we see that $\{b, c\} \notin E$, which is what we wanted to show. ■

- Note that the following similar statement is untrue: "for any graph $G = (V, E)$, if there is a node $v \in V$ where the neighborhood set of v is an independent set, then G is triangle-free". (One counterexample is the graph in part (b).) What part of the above proof relies on the "for any node $v \in V$" part of the statement?

  The proof relies on the fact that the neighborhood set of a, one of the nodes arbitrarily chosen by the reader, is an independent set. This node a is not guaranteed to be the same as the node v from the antecedent of the incorrect statement.

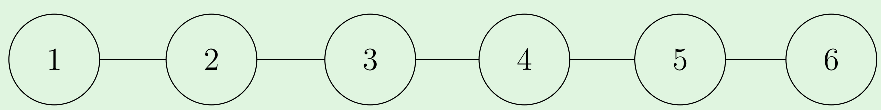
4. Generating Graphs

For the following problems, draw an example of each type of graph.

Many of these have multiple solutions; let me know if you'd like to check your answer!

a. A graph with one connected component, at least 6 nodes, and an independent set with at least 3 elements. Indicate which nodes of your graph are in your independent set.

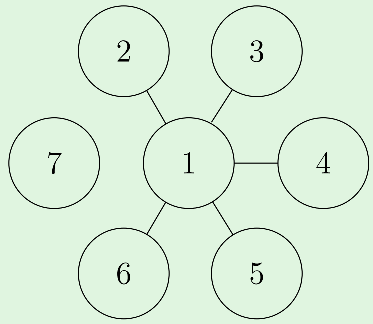
   Here's a graph that could work:

   Two different independent sets in this graph with cardinality 3 would be $\{1, 3, 5\}$ and $\{2, 4, 6\}$. None of these nodes connect to each other.

b. A graph with two connected components, at least 7 nodes, and a vertex cover with 1 element. Indicate which nodes of your graph are in your vertex cover.

Here's a graph that could work:

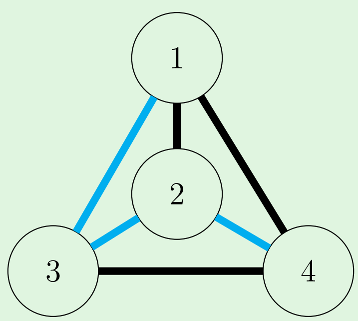


The two connected components of this graph are the component consisting of nodes 1-6, and the component consisting of node 7. (There is no edge between 1 and 7 so that node 7 by itself remains a connected component.)

{1} is a vertex cover of this graph, because each of the edges includes 1 as an endpoint.

c. A copy of K_4 with edges in two colors, so that there is no monochrome copy of K_3. Recall that K_n is our term for the complete graph with n nodes, and a complete graph is a graph where every pair of nodes is connected by an edge.

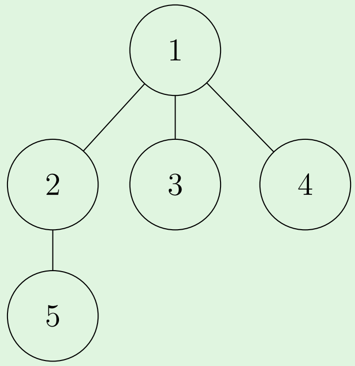
Here's a coloring of the graph that could work. You can come up with this by starting with a non-monochrome copy of K_3, then adding the fourth node and making sure all of the three edges you add are colored so that a monochrome triangle doesn't arise.



Fun fact: 4 is the largest value of n so that K_n is a planar graph, which means a graph that can be drawn without any edges intersecting!

d. A graph with at least 5 nodes where the graph contains no cycles, some node has a degree of at least 3, and a different node has a degree of at least 2.

Here's a graph that could work:



In this graph, node 1 has degree 3 and node 2 has degree 2.