1. Induction Walkthrough: Fibonacci Sums

The **Fibonacci numbers** are a series of numbers that appear in a surprising number of places in math and in the natural world, including in the arrangement of leaves on plants. We can define the Fibonacci numbers as follows:

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

a. Using this definition, write down F_2, F_3, F_4, and F_5.

We'll prove by induction that, for any natural number n, this equality is true:

$$F_0 + F_1 + F_2 + ... + F_n = F_{n+2} - 1$$

b. We'll need some predicate $P(n)$ that we'll show is true for all $n \in \mathbb{N}$. What predicate $P(n)$ should we use? Fill in the first paragraph of the proof.

c. To prove this predicate by induction, we first need to prove a base case. Fill in the second paragraph of the proof. (Hint: How many terms will be in the sum on the left-hand-side?)

d. Next, we need to pick some $k \in \mathbb{N}$, assume that $P(k)$ is true (the inductive hypothesis), then prove that $P(k+1)$ is true. Fill in the third paragraph of the proof.

e. Fill in the equations at the end of the proof. Remember that whenever you're trying to prove that an equation is true, start with one side of the equation and transform it step-by-step until you reach the other side of the equation.

Here is a proof template for you to fill in:

b. Let $P(n)$ be the statement ____________________. We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

c. As our base case, we prove $P($_____$)$, namely that ____________________. To see this, note that $F_0 =$ _____, and $F_2 =$ __________ $=$ _____, so $P($_____$)$ is true, as required.

d. For our inductive step, pick some arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true, meaning that ____________________. We need to show $P(k+1)$, namely that ____________________.

e. To see this, note that

$$\begin{aligned}
\underline{\hspace{6cm}} &= \underline{\hspace{6cm}} \\
&= \underline{\hspace{6cm}} \quad \text{(by our IH)} \\
&= \underline{\hspace{6cm}}.
\end{aligned}$$

We see that ____________________, so $P(k+1)$ is true, completing the induction. ■

2. Inducting Up/Down, Complete Induction

a. Set up the inductive step (assume $P(k)$, prove $P(k+1)$) for these example predicates. Will you "build up", "build down", or neither? What variable(s) will you introduce?

   (1) $P(n)$: "Any graph with n nodes has an even number of nodes with odd degree."

   (2) $P(n)$: "There is a tournament with exactly n players where every player is a champion."

b. How does a proof by complete induction differ from a "normal" proof by induction?

c. Can we always use complete induction instead of "normal" induction? Why or why not? If so, why don't we always use complete induction?

3. Complete Induction: Products of Prime Numbers

A natural number $n > 1$ is **composite** if it can be written as the product of two natural numbers that are both greater than 1. Formally, we'll say n is composite if and only if this statement is true:

$$\exists p \in \mathbb{N}.\ \exists q \in \mathbb{N}.\ (n = pq \land (1 < p < n) \land (1 < q < n))$$

On the other hand, a natural number $n > 1$ is **prime** if it is not composite. For example, 12 is composite: it can be written as $3 \cdot 4$. Meanwhile, 2 is prime, since we can only write it as $2 \cdot 1$.

Using a proof by complete induction, prove this theorem: Every natural number $n > 1$ can be written as a product of one or more prime numbers. (Hint: Every prime number can be written as the product of one prime number: itself. You will also want to use cases in your inductive step.)

a. What is the predicate $P(n)$?

b. Which numbers do you want to show $P(n)$ is true for? What base case does this suggest?

c. To use complete induction, what will you assume for the inductive hypothesis, and what will you want to show for the inductive step?

4. Larger Step Sizes: Even Fibonacci Numbers

We will prove the following theorem by induction: For any natural number n, if there is a natural number m where $n = 3m$, then F_n is even.

a. We will use this statement as our predicate $P(n)$: "F_n is even." Which numbers do we want to show $P(n)$ is true for? Give some examples.

b. We will use a base case of proving $P(0)$. What step size should we use to cover all of the numbers in your answer to the previous part? What would we assume as our inductive hypothesis? (There should be an extra constraint on k!) What would we want to show?