# 1. Induction Walkthrough: Fibonacci Sums

The **Fibonacci numbers** are a series of numbers that appear in a surprising number of places in math and in the natural world, including in the arrangement of leaves on plants. We can define the Fibonacci numbers as follows:

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

a. Using this definition, write down $F_2, F_3, F_4$, and $F_5$.

> $F_2 = 0 + 1 = 1$
>
> $F_3 = 1 + 1 = 2$
>
> $F_4 = 1 + 2 = 3$
>
> $F_5 = 2 + 3 = 5$

We'll prove by induction that, for any natural number $n$, this equality is true:

$$F_0 + F_1 + F_2 + ... + F_n = F_{n+2} - 1$$

b. We'll need some predicate $P(n)$ that we'll show is true for all $n \in \mathbb{N}$. What predicate $P(n)$ should we use? Fill in the first paragraph of the proof.

c. To prove this predicate by induction, we first need to prove a base case. Fill in the second paragraph of the proof. (Hint: How many terms will be in the sum on the left-hand-side?)

d. Next, we need to pick some $k \in \mathbb{N}$, assume that $P(k)$ is true (the inductive hypothesis), then prove that $P(k+1)$ is true. Fill in the third paragraph of the proof.

e. Fill in the equations at the end of the proof. Remember that whenever you're trying to prove that an equation is true, start with one side of the equation and transform it step-by-step until you reach the other side of the equation.

Here is a proof template for you to fill in:

> b. Let $P(n)$ be the statement $\underline{F_0 + F_1 + F_2 + ... + F_n = F_{n+2} - 1}$. We will prove by induction that $P(n)$ holds for all natural numbers $n$, from which the theorem follows.
>
> c. As our base case, we prove $P(\underline{0})$, namely that $\underline{F_0 = F_2 - 1}$. To see this, note that $F_0 = \underline{0}$ by definition, and $F_2 = \underline{F_0 + F_1} = \underline{1}$, so $P(0)$ is true, as required.
>
> d. For our inductive step, pick some arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true, meaning that $\underline{F_0 + F_1 + F_2 + ... + F_k = F_{k+2} - 1}$. We need to show $P(k+1)$, namely that $\underline{F_0 + F_1 + F_2 + ... + F_{k+1} = F_{k+3} - 1}$.
>
> *There are two ways to approach the equations in part (e). We can start from the left-hand*

*side of what we want to show and manipulate the expression to arrive at the right-hand side:*

e. To see this, note that

$$\begin{aligned} \underline{F_0 + F_1 + F_2 + ... + F_k + F_{k+1}} &= \underline{(F_{k+2} - 1) + F_{k+1}} \quad \text{(by our IH)} \\ &= \underline{F_{k+1} + F_{k+2} - 1} \\ &= \underline{F_{k+3} - 1}. \end{aligned}$$

*We can also start from the right-hand side of what we want to show and arrive at the left-hand side:*

e. To see this, note that

$$\begin{aligned} \underline{F_{k+3} - 1} &= \underline{F_{k+1} + F_{k+2} - 1} \\ &= \underline{F_{k+1} + F_0 + F_1 + ... + F_k} \quad \text{(by our IH)} \\ &= \underline{F_0 + F_1 + ... + F_k + F_{k+1}}. \end{aligned}$$

*Either way, we can restate $P(k+1)$ in our last paragraph like so:*

We see that $\underline{F_0 + F_1 + F_2 + ... + F_{k+1} = F_{k+3} - 1}$, so $P(k+1)$ is true, completing the induction. ■

# 2. Inducting Up/Down, Complete Induction

a. Set up the inductive step (assume $P(k)$, prove $P(k+1)$) for these example predicates. Will you "build up", "build down", or neither? What variable(s) will you introduce?

(1) $P(n)$: "Any graph with $n$ nodes has an even number of nodes with odd degree."

> We will assume $P(k)$: any graph with $k$ nodes has an even number of nodes with odd degree. We want to show $P(k+1)$: any graph with $k+1$ nodes has an even number of nodes with odd degree.
>
> This predicate is a universal statement and we will "build down". To prove the universal statement $P(k+1)$, we should start by asking the reader to pick an arbitrary graph with $k+1$ nodes, then show that it has an even number of nodes with an odd degree.

(2) $P(n)$: "There is a tournament with exactly $n$ players where every player is a champion."

> We will assume $P(k)$: there is a tournament with exactly $k$ players where every player is a champion. We want to show $P(k+1)$: there is a tournament with exactly $k+1$ players where every player is a champion.
>
> This predicate is an existential statement and we will "build up". Because we are assuming the existential statement $P(k)$, we know there is a tournament with the provided properties that has exactly $k$ players. We are trying to find a tournament with exactly $k+1$ players, so in our inductive step, we can introduce the graph from the inductive hypothesis and describe how to modify it to produce a new tournament with $k+1$ nodes.
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> (Note that this predicate is actually only true for odd natural numbers, so we would actually be trying to prove $P(k+2)$.)

b. How does a proof by complete induction differ from a "normal" proof by induction?

> In "normal" induction, our inductive hypothesis is just assuming that $P(k)$ is true.
>
> In contrast, in a proof by complete induction, our inductive hypothesis will have more to it. Instead of just assuming $P(k)$ is true, we assume our predicate is true for all natural numbers between our base case and $k$. In other words, when we are trying to prove that $P(n)$ is true for all natural numbers $n > b$, we assume $P(b), P(b+1), \ldots.P(k-1), P(k)$.

c. Can we always use complete induction instead of "normal" induction? Why or why not? If so, why don't we always use complete induction?

We can always use complete induction anywhere we would use "normal" induction. This is because a proof by "normal" induction proves the inductive step based on the assumption that $P(k)$ is true. We do still assume $P(k)$ in complete induction, so the proof can still proceed like it does in "normal" induction.

We don't always use complete induction because we don't always need the additional assumptions that complete induction introduces. We try not to introduce anything into a proof that we don't end up using.

# 3. Complete Induction: Products of Prime Numbers

A natural number $n > 1$ is **composite** if it can be written as the product of two natural numbers that are both greater than 1. Formally, we'll say $n$ is composite if and only if this statement is true:

$$\exists p \in \mathbb{N}.\ \exists q \in \mathbb{N}.\ (n = pq \wedge (1 < p < n) \wedge (1 < q < n))$$

On the other hand, a natural number $n > 1$ is **prime** if it is not composite. For example, 12 is composite: it can be written as $3 \cdot 4$. Meanwhile, 2 is prime, since we can only write it as $2 \cdot 1$.

Using a proof by complete induction, prove this theorem: Every natural number $n > 1$ can be written as a product of one or more prime numbers. (Hint: Every prime number can be written as the product of one prime number: itself. You will also want to use cases in your inductive step.)

a. What is the predicate $P(n)$?

> We can have $P(n)$ be the predicate "$n$ can be written as a product of one or more prime numbers."

b. Which numbers do you want to show $P(n)$ is true for? What base case does this suggest?

> We want to show that $P(n)$ is true for all natural numbers greater than 1, which is the same as saying all natural numbers greater than or equal to 2, so we could use a base case of proving $P(2)$.

c. To use complete induction, what will you assume for the inductive hypothesis, and what will you want to show for the inductive step?

> For the inductive hypothesis, the reader will pick $k$, and we will assume that the predicate is true for all values up through $P(k)$. Notice that instead of saying $P(0), P(1), ..., P(k)$ are all true, since our base case starts at $P(2)$, we will assume that $P(2), P(3), ..., P(k)$ are all true. This means that we're assuming 2 can be written as the product of one or more prime numbers, 3 can be written as the product of one or more prime numbers, 4 can be, and so on, up to $k$ being written as the product of one or more prime numbers.
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> We will then show that $P(k + 1)$ is true. In other words, we want to show that $k + 1$ can be written as the product or one or more prime numbers.

**Proof:** Let $P(n)$ be the statement "$n$ can be written as the product of one or more prime numbers." We will prove by complete induction that $P(n)$ holds for all $n > 1$, from which the theorem follows.

For the base case, we need to show $P(2)$, that 2 can be written as the product of one or more primes. We can express 2 as the product of one prime, namely itself.

For the inductive step, pick some arbitrary $k > 2$ and assume that $P(2), P(3), ..., P(k)$ hold. We will prove $P(k+1)$, namely that $k+1$ can be written as the product of one or more primes.

To do so, consider two cases. If $k+1$ is prime, then it can be written as the product of just itself, and we're done. Otherwise, $k+1$ is composite, so it can be written as $pq$ for some natural numbers $p$ and $q$ where both $p$ and $q$ are greater than or equal to 2 and less than or equal to $k$. Therefore, our inductive hypothesis applies to $p$ and $q$, we can express $p$ and $q$ each as products of primes. We can then express $k+1$ as a product of primes by multiplying together all the prime numbers in the expressions of $p$ and $q$ as products of prime numbers. Therefore, we see that $P(k+1)$ is true, completing the induction. ■

# 4. Larger Step Sizes: Even Fibonacci Numbers

We will prove the following theorem by induction: For any natural number $n$, if there is a natural number $m$ where $n = 3m$, then $F_n$ is even.

a. We will use this statement as our predicate $P(n)$: "$F_n$ is even." Which numbers do we want to show $P(n)$ is true for? Give some examples.

> We want to show that $P(n)$ is true for all natural numbers that are multiples of 3: so 0, 3, 6, 9, and so on.

b. We will use a base case of proving $P(0)$. What step size should we use to cover all of the numbers in your answer to the previous part? What would we assume as our inductive hypothesis? (There should be an extra constraint on $k$!) What would we want to show?

> We would need a step size of 3.
>
> The reader picks any $k$ where there is a natural number $m$ so that $k = 3m$. We would assume $P(k)$ is true and show that $P(k + 3)$ is true.

**Proof:** Let $P(n)$ be the statement "$F_{3n}$ is even." We will prove by complete induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

For the base case, we need to show $P(0)$, that $F_0$ is even. By definition, $F_0$ is equal to 0. 0 is an even number because there exists an integer $k$ (namely 0) where $2k = 0$.

For the inductive step, pick some arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true, i.e. that $F_{3k}$ is even. We will prove $P(k + 1)$, i.e. that $F_{3k+3}$ is even.

To do so, notice that

$$\begin{aligned} F_{3k+3} &= F_{3k+2} + F_{3k+1} \\ &= F_{3k+1} + F_{3k} + F_{3k+1} \\ &= 2F_{3k+1} + F_{3k}. \end{aligned}$$

The number $2F_{3k+1}$ is even because it is twice another integer (namely $F_{3k+1}$). By our inductive hypothesis, $F_{3k}$ is even. ■