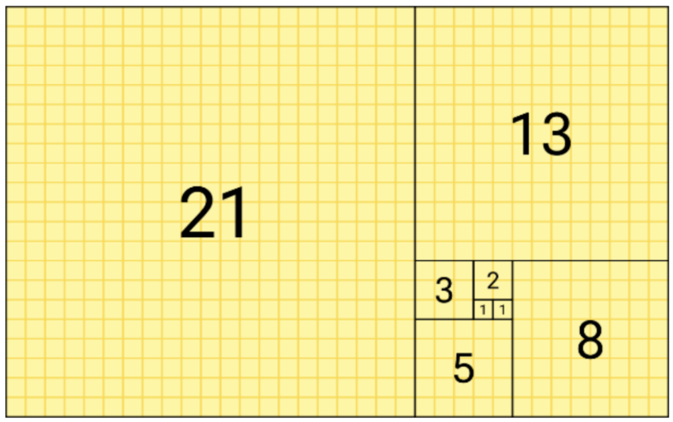
1. Recurrence Relations: More Fibonacci Sums

The sums of squares of Fibonacci numbers also have a bunch of cool properties. Specifically, for any $n \in \mathbb{N}$, the following statement is true:

$$F_0^2 + F_1^2 + \dots + F_n^2 = F_n F_{n+1}.$$

Here's a graphical intuition for where this comes from:



Prove this statement by induction.

a. What is the predicate $P(n)$ we should use?

b. What natural numbers are we trying to prove $P(n)$ for? What base case and step size does this suggest?

c. What will you assume as the inductive hypothesis and want to show for the inductive step?

2. Induction with Larger Step Sizes: Socks in a Box

Consider this game for two players, which we will call the n-Sock Game. Begin with a box with n socks in it. The first player takes out between 1 and 10 socks. Then the second player takes out between 1 and 10 socks. This process repeats until the box is empty. At that point, the player who has the next turn loses, since they can't take out between 1 and 10 socks, and the other player wins.

Prove this theorem by induction: For any natural number n that is a multiple of 11, there is a strategy that the second player can use to always win the n-Sock Game. The multiples of 11 are the numbers 0, 11, 22, 33, and so on. Answer these questions before starting:

a. How can we convert this theorem into a statement that some predicate $P(n)$ is true for some natural numbers? What is the predicate $P(n)$ and what numbers do we want to show $P(n)$ is true for? What base case and step size does this suggest?

b. Are we "inducting up" or "inducting down"? What will you assume and want to show for the inductive step?

3. Induction with Multiple Variables: Factorials!

Given a natural number n, the notation $n!$, pronounced "n factorial", represents the product $1 \cdot 2 \cdot \ldots \cdot n$. Formally, we can define $n!$ using a recurrence relation:

$$0! = 1 \qquad (n+1)! = (n+1) \cdot n!$$

We'll prove the following theorem by induction: For any $m, n \in \mathbb{N}$, we have that $(m!)(n!) \leq (m+n)!$. To do this, we'll let $P(n)$ be the statement "for any $m \in \mathbb{N}$, we have that $(m!)(n!) \leq (m+n)!$".

a. Explain why proving that $P(n)$ is true for any $n \in \mathbb{N}$ is the same as proving the theorem.

b. Explain why including "for any $m \in \mathbb{N}$" in the statement of $P(n)$ does not violate the Induction Proofwriting Checklist item on variable scoping.

c. What natural numbers do we need to prove $P(n)$ for? What base case and step size does this suggest we use?

d. Are we "inducting up" or "inducting down"? What will you assume as the inductive hypothesis and want to show for the inductive step?