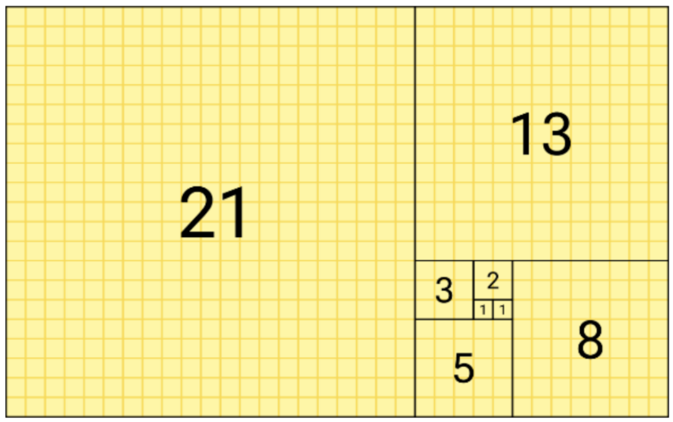
1. Recurrence Relations: More Fibonacci Sums

The sums of squares of Fibonacci numbers also have a bunch of cool properties. Specifically, for any $n \in \mathbb{N}$, the following statement is true:

$$F_0^2 + F_1^2 + ... + F_n^2 = F_n F_{n+1}.$$

Here's a graphical intuition for where this comes from:



Prove this statement by induction.

a. What is the predicate $P(n)$ we should use?

The predicate $P(n)$ should be the equation in the problem statement: $F_0^2 + F_1^2 + ... + F_n^2 = F_n F_{n+1}$.

b. What natural numbers are we trying to prove $P(n)$ for? What base case and step size does this suggest?

Since we are trying to prove $P(n)$ for all natural numbers, a base case of 0 and step size of 1 is a good starting point.

c. What will you assume as the inductive hypothesis and want to show for the inductive step?

We'll assume $P(k)$, which says that

$$F_0^2 + F_1^2 + ... + F_k^2 = F_k F_{k+1}.$$

We want to show $P(k+1)$, which says that

$$F_0^2 + F_1^2 + ... + F_{k+1}^2 = F_{k+1} F_{k+2}.$$

Proof: Let $P(n)$ be the statement "$F_0^2 + F_1^2 + ... + F_n^2 = F_n F_{n+1}$." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

As our base case, we will prove $P(0)$, that $F_0^2 = F_0 F_1$. To see this, note that $F_0 = 0$, so F_0^2 $= 0$ and $F_0 F_1 = 0$. Therefore, we see that $F_0^2 = F_0 F_1$ as required.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that $F_0^2 + F_1^2 + ... + F_k^2 = F_k F_{k+1}$. We need to prove that $P(k+1)$ is true, meaning that we need to show $F_0^2 + F_1^2 + ... + F_{k+1}^2 = F_{k+1} F_{k+2}$.

To see this, notice that

$$\begin{aligned} F_0^2 + F_1^2 + ... + F_{k+1}^2 &= (F_0^2 + F_1^2 + ... + F_k^2) + F_{k+1}^2 \\ &= F_k F_{k+1} + F_{k+1}^2 \text{ (by our IH)} \\ &= (F_{k+1})(F_k + F_{k+1}) \\ &= F_{k+1} F_{k+2}. \end{aligned}$$

Therefore, we see that $P(k+1)$ is true, as required.

2. Induction with Larger Step Sizes: Socks in a Box

Consider this game for two players, which we will call the n-Sock Game. Begin with a box with n socks in it. The first player takes out between 1 and 10 socks. Then the second player takes out between 1 and 10 socks. This process repeats until the box is empty. At that point, the player who has the next turn loses, since they can't take out between 1 and 10 socks, and the other player wins.

Prove this theorem by induction: For any natural number n that is a multiple of 11, there is a strategy that the second player can use to always win the n-Sock Game. The multiples of 11 are the numbers 0, 11, 22, 33, and so on. Answer these questions before starting:

a. How can we convert this theorem into a statement that some predicate $P(n)$ is true for some natural numbers? What is the predicate $P(n)$ and what numbers do we want to show $P(n)$ is true for? What base case and step size does this suggest?

> We can have $P(n)$ be the predicate "there is a strategy that the second player can use to always win the n-Sock Game". We want to show that $P(n)$ is true for all natural numbers that are a multiple of 11, so we can start with a base case of $P(0)$ and prove $P(k+11)$ in our inductive step. This form of the predicate is what I'll be using for the rest of the proof.
>
> Alternatively, we could have $P(n)$ be the predicate "there is a strategy that the second player can use to always win the $11n$-Sock Game". Then, we would want to show that $P(n)$ is true for all natural numbers, so we can start with a base case of $P(0)$ and prove $P(k+1)$ in our inductive step.

b. Are we "inducting up" or "inducting down"? What will you assume and want to show for the inductive step?

> In the inductive step, we will assume $P(k)$ is true and show that $P(k+11)$ is true.
>
> Since $P(n)$ is an existentially quantified statement – "there is a strategy" – we will want to build up. That is, we will start by assuming there is a winning strategy for the k-Sock Game and try to get from that strategy to a winning strategy for the $k+11$-Sock Game. Specifically, it turns out that there's a specific strategy that works, so we'll modify it to specify what the strategy is that we need to follow.

Proof: Let $P(n)$ be the statement "there is a strategy that the second player can use to always win the n-Sock Game". We will show by induction that $P(n)$ is true for all natural numbers that are a multiple of 11, from which the theorem follows.

For our base case, we will show $P(0)$, that there is a strategy that the second player can use so that there are 0 socks in the box before the first player's turn. Notice that no matter what the first player does, the box will have 0 socks before the first player's first turn, so the second player will always win the 0-Sock Game.

For our inductive step, pick an arbitrary $k \in \mathbb{N}$ where k is a multiple of 11. Assume $P(k)$, namely that the second player has a strategy to win the k-Sock Game. We'll show $P(k+11)$, namely that the second player has a strategy to win the $(k+11)$-Sock Game.

This strategy is as follows: the second player uses their strategy to win the k-Sock Game. After following this strategy, the box will be left with 11 socks in it before one of the first player's turns. Then, say the first player takes out n socks from the box, where $1 \leq n \leq 10$ by the rules of the game, leaving the box with $11 - n$ socks in it. The second player can then remove $11 - n$ socks, leaving the box empty. Since $1 \leq n \leq 10$, we see that $1 \leq 11 - n \leq 10$, so this is a valid move. Because this placement leaves the box empty prior to the first player's next turn, the first player loses and the second player wins. Then, $P(k+1)$ holds, completing the induction. ■

3. Induction with Multiple Variables: Factorials!

Given a natural number n, the notation $n!$, pronounced "n factorial", represents the product $1 \cdot 2 \cdot \ldots \cdot n$. Formally, we can define $n!$ using a recurrence relation:

$$0! = 1 \qquad (n+1)! = (n+1) \cdot n!$$

We'll prove the following theorem by induction: For any $m, n \in \mathbb{N}$, we have that $(m!)(n!) \leq (m+n)!$. To do this, we'll let $P(n)$ be the statement "for any $m \in \mathbb{N}$, we have that $(m!)(n!) \leq (m+n)!$".

a. Explain why proving that $P(n)$ is true for any $n \in \mathbb{N}$ is the same as proving the theorem.

> If $P(n)$ is true for all natural numbers n, then it means that for any choice of n, the following is true: for any choice of m, we have $m!n! \leq (m+n)!$. This is the same as what's stated in the theorem.

b. Explain why including "for any $m \in \mathbb{N}$" in the statement of $P(n)$ does not violate the Induction Proofwriting Checklist item on variable scoping.

> The proofwriting checklist item only applies to the variable that P is specified around. In this case, since we're using $P(n)$, it would be wrong to include "for all $n \in \mathbb{N}$". But m is a separate variable, which does need to be introduced before we can use it, so it's okay to include a quantifier for it.

c. What natural numbers do we need to prove $P(n)$ for? What base case and step size does this suggest we use?

> Since we want to prove that $P(n)$ is true for any natural number n, we can start at $P(0)$ and use a step size of 1.

d. Are we "inducting up" or "inducting down"? What will you assume as the inductive hypothesis and want to show for the inductive step?

> Because we're using a step size of 1, we'll assume $P(k)$ and prove $P(k+1)$.
>
> Since $P(n)$ is a universally quantified statement, we will be "inducting down": since we want to prove that for all natural numbers m, the equation with $k+1$ holds, we will pick an arbitrary natural number m, then try to find a way to transform the expression involving $k+1$ into the expression involving k so that we can apply our inductive hypothesis.

Proof: Let $P(n)$ be the statement "for any $m \in \mathbb{N}$, we have that $m!n! \leq (m+n)!$ is true." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we prove $P(0)$, namely that for any $m \in \mathbb{N}$, we have $m!0! \leq (m+0)!$. Pick a natural number m, and notice that

$$m!0! = m! \cdot 1 = m! = (m+0)!$$

Therefore, we know that $m!0! \leq (m+0)!$, as required.

For our inductive step, pick a $k \in \mathbb{N}$ and assume that $P(k)$ holds, so for any $m \in \mathbb{N}$, we know that $m!k! \leq (m+k)!$. We will prove $P(k+1)$: for any $m \in \mathbb{N}$, we have $m!(k+1)! \leq (m+k+1)!$. To do this, start by picking an $m \in \mathbb{N}$. Then we can see that:

$$\begin{aligned} m!(k+1)! &= m!k!(k+1) \\ &\leq (m+k)!(k+1) && \text{(by the IH)} \\ &< (m+k)!(m+k+1) \\ &= (m+k+1)! \end{aligned}$$

This tells us that $m!(k+1)! \leq (m+k+1)!$, so $P(k+1)$ holds, completing the induction. ■