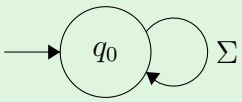
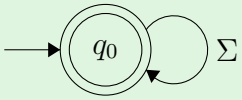
1. Basic DFAs

Let $\Sigma = \{\texttt{a}, \texttt{b}\}$. Draw a DFA for the following languages over Σ. The DFA must accept every string in the language and reject every string not in the language.

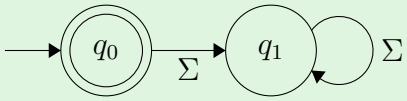
a. $\varnothing$



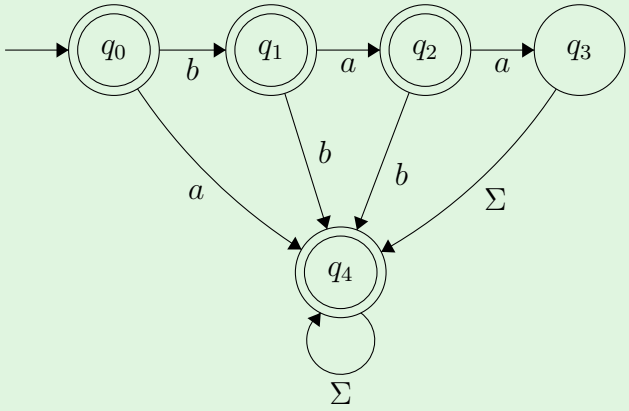
b. Σ^*



c. $\{\varepsilon\}$



d. $\overline{L}$, where $L = \{\texttt{baa}\}$ (Hint: First write out a DFA for L, then take the complement.)



2. DFAs, States, and Memory

One of the best starting points for designing a DFA is that the DFA's only "memory" is the state it's in at any given point. When designing DFAs, you can make every state correspond to some piece of information the automaton must "remember" about the string it's seen so far.

Consider this language over $\Sigma = \{\texttt{a}, \texttt{b}\}$, which we'll call L:

$$\{w \in \Sigma^* \mid w \text{ has an odd number of } a\text{'s and a number of } b\text{'s that is a multiple of 3}\}$$

a. When doing a DFA design problem, it's a good strategy to come up with strings to test a DFA on. Write down three strings from Σ^* that are in L and three strings from Σ^* that are not in L.

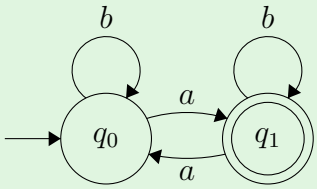
Some strings that are in L are `a`, `bbbaaa`, and `aabababa`.

Some strings that are not in L are ε, `b`, and `abb`.

b. Another starting point is to work on a simpler version of the problem. Draw out a DFA for this simpler language:

$$\{w \in \Sigma^* \mid w \text{ has an odd number of } a\text{'s}\}$$

Then, think about what information each state represents.

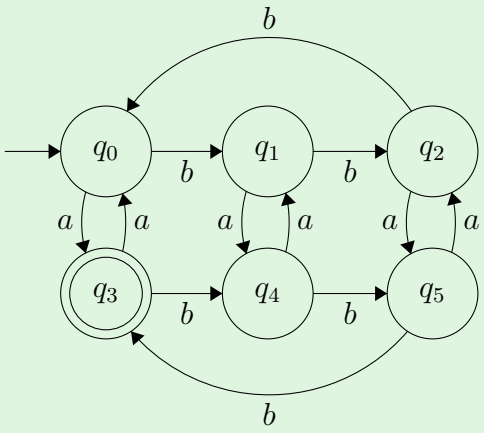


The state q_0 represents the string so far having an even number of a's. The state q_1 represents the string so far having an odd number of a's. If we see a b, it doesn't change our state. If we see an a, we'll swap to the other state.

c. What pieces of information would a DFA for L need to know/remember about the string that it's seen so far? Hint: think about the DFA from class for the language of strings involving a modular congruence. You should have six possible states.

A DFA for L would need to know whether the number of a's so far is even or odd, as well as whether the number of b's is a multiple of 3, one more than a multiple of 3, or two more than a multiple of 3. Since we have two options for the first choice and three options for the second choice, we have six states.

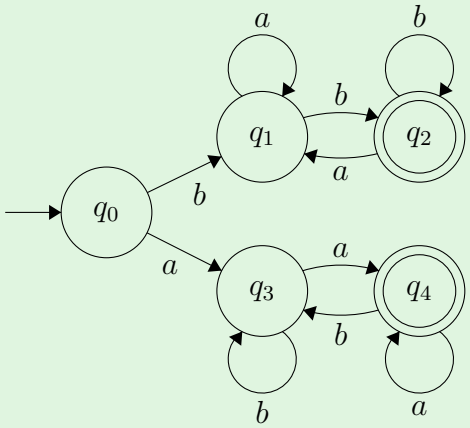
d. Now, add one state for each piece of information the DFA would need to remember. Then, add in transitions based on how the information changes upon seeing one more character. Then, mark the appropriate states as accepting or not.



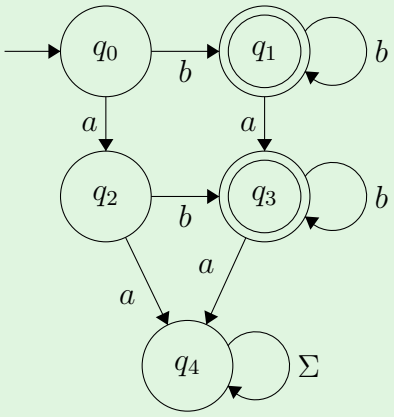
Here, the top row of states represents having seen an even number of a's, and the bottom row of states represents having seen an odd number of a's. The first column of states represents having a number of b's that is a multiple of 3, the middle column represents a number of b's that is 1 more than a multiple of 3, and the final column represents a number of b's that is 2 more than a multiple of 3.

Here are some additional languages you can practice building DFAs for:

e. $\{w \in \Sigma^* \mid |w| > 1$ and the first and last character of w are the same $\}$



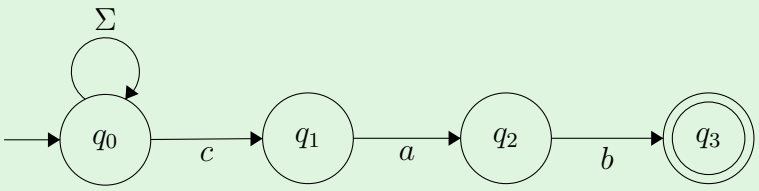
f. The set of strings with at least one `b`, but no more than one `a`.


q_0
q_1
q_2
q_3
q_4
a
b
Σ

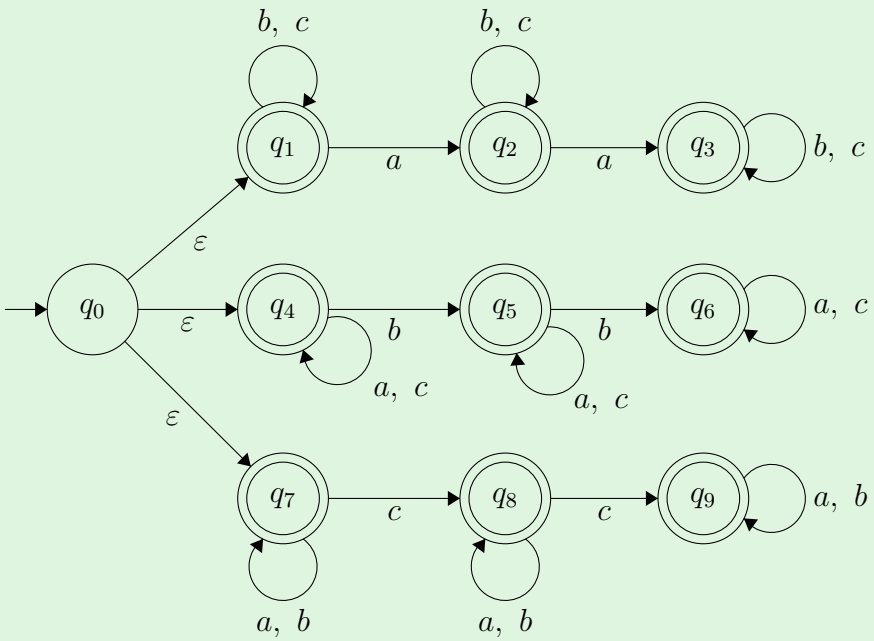
3. Building NFAs

Design an NFA for these languages:

a. $L = \{w \in \Sigma^* \mid w \text{ ends in } \texttt{cab}\}$ over $\Sigma = \{\texttt{a}, \texttt{b}, \texttt{c}\}$.



b. $L = \{w \in \Sigma^* \mid$ there is a character in Σ that appears at most twice in $w\}$, over $\Sigma = \{\texttt{a}, \texttt{b}, \texttt{c}\}$. You'll want to use the guess-and-check model here, since any DFA for this language will require at least 64 states! (Hint: First try doing this with a smaller alphabet like $\{\texttt{a}, \texttt{b}\}$. What would you do if you knew what character was going to appear at most twice?)



4. Kleene Star and Language Concatenation Practice

Let's practice applying the formal definitions of language concatenation and the Kleene star:

$$L_1L_2 = \{w \mid \exists x \in L_1.\ \exists y \in L_2.\ w = xy\}$$

$$L^0 = \{\varepsilon\} \qquad L^{n+1} = LL^n$$

$$L^* = \{w \mid \exists n \in \mathbb{N}.\ w \in L^n\}$$

a. Let $L = \{\texttt{abb}, \texttt{c}\}$. What is L^2? Give three examples of strings in L^*. Is $\varepsilon \in L^*$?

L^2 is the set $\{\texttt{abbc}, \texttt{cabb}, \texttt{abbabb}, \texttt{cc}\}$.

Some examples of strings in L^* would be `abbabbcabbcc`, `abb`, and `cccc`.

Yes, ε is in L^*. This is true no matter what language L is.

b. Let $L = \{\varepsilon\}$. What is L^*?

Intuitively, we can't make any other strings when we put together the empty string with itself, no matter how many copies of the empty string we use. Formally, $\{\varepsilon\}$ concatenated with any language produces that language again. That means that L^* is $\{\varepsilon\}$.

c. Let $L = \varnothing$. What is L^*?

Intuitively, if you take no strings from the empty set and concatenate them, you get the empty string. Therefore, $\varnothing^0$ still contains the empty string and $\varnothing^*$ also contains the empty string.

Formally, since L^0 is $\{\varepsilon\}$ for any language L, including $\varnothing$, we can see that $\varnothing^*$ is still $\{\varepsilon\}$.

d. Let $\Sigma = \{\texttt{a}, \texttt{b}\}$. Let L be the language $\{w \in \Sigma^* \mid |w| \not\equiv_3 2\}$. What is L^2? (Hint: Does this look like something from Problem Set 3?)

L^2 is the set containing all strings over a and b, namely Σ^*.

5. Proofs on Languages: The Finite Power Property

Since languages are sets of strings, proofs about languages will look similar to the proofs we wrote about sets. In particular, be mindful of the formal definitions listed earlier!

a. Let L be a language. Prove that for any natural number n, $L^n \subseteq L^*$. While L^n and L^* are languages, this is fundamentally the same as any subset problem we've previously tackled!

Proof: Pick an arbitrary natural number. We'll prove that $L^n \subseteq L^*$. To do so, let w be an arbitrary string in L^n. This string is in L^* because n is a natural number such that w is in L^n. ■

b. We say that a language L has the **finite power property** if the following statement is true:

$$\exists k \in \mathbb{N}.\ L^k = L^{k+1}$$

Come up with two languages that have the finite power property.

Some examples are $\{\varepsilon\}$ and $\{\varepsilon, a, aa, aaa, aaaa, \dots\}$

c. Let L be a language with the finite power property, and let k be a natural number where $L^k = L^{k+1}$. Prove by induction that for all natural numbers $n \geq k$, $L^k = L^n$.

Proof: Let $P(n)$ be the predicate $L^k = L^n$. We'll prove by induction that $P(n)$ is true for all natural numbers $n \geq k$.

For our base case, we'll prove $P(k)$. It is clear that $L^k = L^k$, so $P(k)$ holds.

For our inductive step, pick an arbitrary natural number $m \geq k$ and assume that $P(m)$ holds. We'll prove $P(m+1)$, namely that $L^k = L^{m+1}$. To do so, notice that L^{m+1} is the same as LL^m, which by our inductive hypothesis is the same as LL^k. Since we know that k is a number such that $L^k = L^{k+1}$, we can finally see that $LL^k = L^k$, so $P(m+1)$ holds, as required. ■

This problem is from Wojciech Rytter's "100 Exercises in the Theory of Automata and Formal Languages".